Solutions: Problem Set 1

2-16, 18, 32, 36

2-16

$$(r, \phi, z) = \left(4, \frac{2\pi}{3}, 3\right)$$

a) in Cartesian coords?

$$z = z$$

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$x = 4 \cdot \cos\left(\frac{2\pi}{3}\right) = 4 \cdot \left(-\frac{1}{2}\right) = -2$$

$$y = 4 \cdot \sin\left(\frac{2\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$= 3.4641$$

$$z = 3$$

$$(x, y, z) = (-2, 2\sqrt{3}, 3)$$

b) in spherical coords?

note: in cyl. $x = r\cos\phi \longrightarrow x^2 = r^2\cos^2\phi$
$y = r\sin\phi \longrightarrow y^2 = r^2\sin^2\phi$ } add

$\hookrightarrow r^2 = x^2 + y^2$

In spher., $R = \sqrt{x^2+y^2+z^2}$

$$= \sqrt{r^2 + z^2}$$

$$R = \sqrt{4^2+3^2} = 5$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{r}{z} = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$$

$$\phi_{sph} = \phi_{cyl} = \frac{2\pi}{3} = 120^\circ$$

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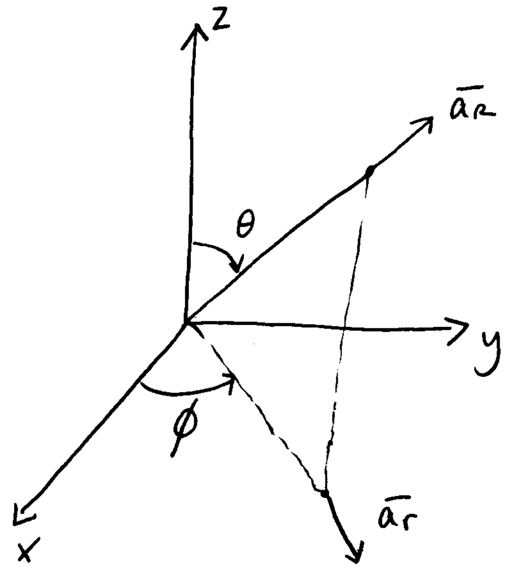
$$(R, \theta, \phi) = (5, 53.1^\circ, 120^\circ)$$

2-18

We want $\bar{a}_R, \bar{a}_\theta, \bar{a}_\phi$ in terms of $\bar{a}_x, \bar{a}_y, \bar{a}_z$

$\bar{a}_R$

$$\bar{a}_R = \alpha_1 \bar{a}_x + \alpha_2 \bar{a}_y + \alpha_3 \bar{a}_z$$



To simplify, put $\bar{a}_R$ in terms of cylindrical components

$$\bar{a}_R = \beta_1 \bar{a}_r + \beta_2 \cancel{\bar{a}_\phi}^{\,0} + \alpha_3 \bar{a}_z$$

$$\bar{a}_R = \sin\theta\, \bar{a}_r + \cos\theta\, \bar{a}_z$$

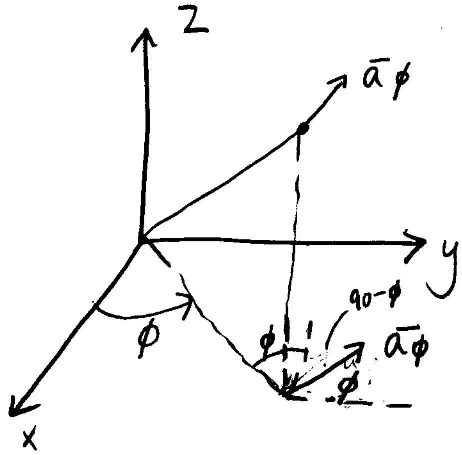
$$\bar{a}_r = \gamma_1 \bar{a}_x + \gamma_2 \bar{a}_y$$

$$\bar{a}_r = \cos\phi\, \bar{a}_x + \sin\phi\, \bar{a}_y$$

$$\bar{a}_R = \sin\theta(\cos\phi\, \bar{a}_x + \sin\phi\, \bar{a}_y) + \cos\theta\, \bar{a}_z$$

$$\bar{a}_R = \sin\theta\cos\phi\, \bar{a}_x + \sin\theta\sin\phi\, \bar{a}_y + \cos\theta\, \bar{a}_z$$

$\bar{a}_\phi$



· independent and orthogonal to $\bar{z}$ so $a_z = 0$

$$\bar{a}_\phi = \alpha_1 \bar{a}_x + \alpha_1 \bar{a}_y$$

Easiest to do as if cylindrical since a_z does not matter.

$$\bar{a}_\phi = -\sin\phi\, \bar{a}_x + \cos\phi\, \bar{a}_y$$

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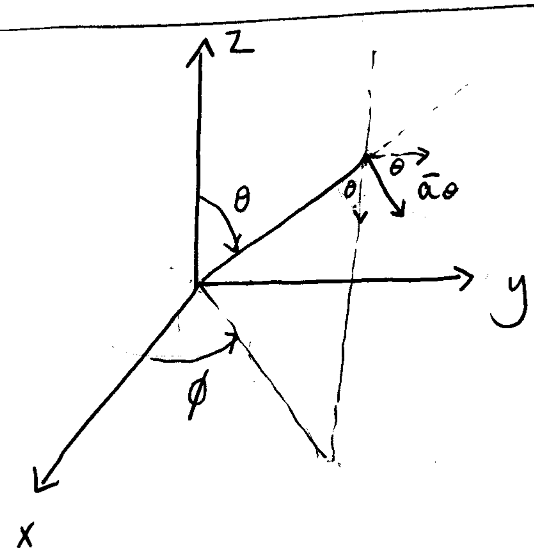
a_θ



$$\bar{a}_\theta = \alpha_1 \bar{a}_x + \alpha_2 \bar{a}_y + \alpha_3 \bar{a}_z$$

$$\bar{a}_\theta = \bar{a}_x(\cos\phi)(\cos\theta) + \bar{a}_y(\sin\phi)(\cos\theta) + \bar{a}_z(-\sin\theta)$$

→ Put them as functions of x, y, z

$$\bar{a}_R = \sin\theta\cos\phi\,\bar{a}_x + \sin\theta\sin\phi\,\bar{a}_y + \cos\theta\,\bar{a}_z$$

$$\sin\theta = \frac{r}{\sqrt{x^2+y^2+z^2}} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

so

$$\boxed{\bar{a}_R = \frac{\bar{a}_x x + \bar{a}_y y + \bar{a}_z z}{\sqrt{x^2+y^2+z^2}}}$$

$$a_\theta = \bar{a}_x(\cos\phi)(\cos\theta) + \bar{a}_y(\sin\phi)(\cos\theta) + \bar{a}_z(-\sin\theta)$$

$$\boxed{\bar{a}_\theta = \frac{xz\,\bar{a}_x + yz\,\bar{a}_y - (x^2+y^2)\,\bar{a}_z}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}}$$

$$\bar{a}_\phi = \frac{-y\,\bar{a}_x + x\,\bar{a}_y}{\sqrt{x^2+y^2}}$$

2-32

$$\bar{D} = \bar{a}_R \,(\cos^2\phi)/R^3$$

Spherical coordinate system.

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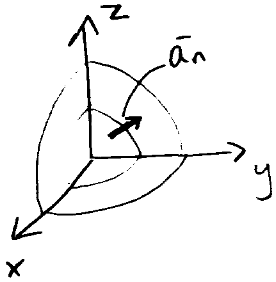

a) $\oint_S \bar{D}\cdot d\bar{s} = \oint \bar{D}\cdot\bar{a}_n d_s$

$\bar{a}_n = \bar{a}_R$
↳ unit normal vector

$$ds = R^2 \sin\theta \, d\theta \, d\phi$$

$$\oint D\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} \left(\bar{a}_R \frac{(\cos^2\phi)}{\cancel{R^3}}\right)\cdot \bar{a}_R \cdot \cancel{R^2}\sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{2}-\frac{1}{1}\right)\sin\theta\, d\theta \;\cos^2\phi\, d\phi$$

$$= -\frac{1}{2}\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi}\cos^2\phi\, d\phi$$

$$= -\frac{1}{2}\Big[-\cos\theta\Big]_0^{\pi}\left[\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right]_0^{2\pi}$$

$$= -\frac{1}{\cancel{2}}\left[\cancel{1+1}\right]\left[\frac{1}{\cancel{2}}\cancel{2}\pi\right] = -\pi$$

$$\oint \bar{D}\cdot d\bar{s} = -\pi$$

b) $\int \nabla \cdot \bar{D} \, dv = ?$

$$dv = R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$\bar{D} = \bar{a}_R \left(\frac{\cos^2\phi}{R^3} \right) \longrightarrow D_R = \frac{\cos^2\phi}{R^3}, \quad D_\phi = 0, \; D_\theta = 0$$

$$\nabla \cdot \bar{D} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 D_R) + \frac{1}{R\sin\theta} \frac{\partial}{\partial \theta} \cancel{(D_\theta \sin\theta)}^{0} + \frac{1}{R\sin\theta} \cancel{\frac{\partial D_\phi}{\partial \phi}}^{0}$$

$$\nabla \cdot \bar{D} = \frac{1}{R^2} \frac{\partial}{\partial R} \left(\cancel{R^2} \frac{\cos^2(\phi)}{R^{\cancel{3}}} \right) = \frac{1}{R^2} \left(-\frac{1}{R^2} \right) \cos^2\phi$$

$$\nabla \cdot \bar{D} = -\frac{1}{R^4} \cos^2\phi$$

$$\int \nabla \cdot \bar{D} \, dv = \int_0^{2\pi} \int_0^{\pi} \int_1^{2} \left(-\frac{1}{\cancel{R^4}_{R^2}} \cos^2\phi \right) \cancel{R^2} \sin\theta \, dR \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi} \sin\theta \, d\theta \int_1^{2} \left(-\frac{1}{R^2} \right) dR$$

$$= \left[\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi \right]_0^{2\pi} \left[-\cos\theta \right]_0^{\pi} \left[\frac{1}{R} \right]_1^{2}$$

$$= \left[\frac{1}{\cancel{2}} \cancel{2}\pi \right] \left[1 + 1 \right] \left[\frac{1}{2} - 1 \right] = -\pi$$

$$\underline{\int \nabla \cdot \bar{D} \, dV = -\pi}$$

2-36 $\bar{A} = \bar{a}_\phi \sin(\phi/2)$ 8

$A_\phi = \sin\frac{\phi}{2}$

Stokes theorem:

$$\int_S (\nabla\times\bar{A})\cdot d\bar{s} = \oint A\cdot d\bar{l}$$

$\bar{a}_n = \bar{a}_R$

$$\nabla\times\bar{A} = \bar{a}_R \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \cancel{\frac{\partial A_\theta}{\partial\phi}}\right]$$

$$+\bar{a}_\theta \frac{1}{R}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_R}{\partial\phi}} - \frac{\partial}{\partial R}(RA_\phi)\right]$$

$$+\bar{a}_\phi \frac{1}{R}\left[\frac{\partial}{\partial R}(R\cancel{A_\theta}) - \cancel{\frac{\partial A_R}{\partial\theta}}\right]$$

$$\nabla\times\bar{A} = \bar{a}_R \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\sin(\tfrac{\phi}{2})\cdot\sin\theta\right)$$

$$-\bar{a}_\theta \frac{1}{R}\frac{\partial}{\partial R} R\sin\frac{\phi}{2}$$

$$\nabla\times\bar{A} = \bar{a}_R \frac{1}{R\sin\theta}\cdot\sin\frac{\phi}{2}\cos\theta - \bar{a}_\theta \frac{1}{R}\sin\frac{\phi}{2}$$

$$\int_S \nabla\times\bar{A}\cdot ds = ?$$

$$ds = R^2 \sin\theta \, d\theta \, d\phi$$

$$d\bar{s} = \bar{a}_R \, ds$$

$$\int_S \nabla\times\bar{A}\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\cancel{R\sin\theta}} \cos\theta \sin\frac{\phi}{2} \cdot R^{\cancel{2}} \cancel{\sin\theta}\, d\theta\, d\phi$$ (9)

R is constant $\rightarrow$ $R = b$

$$= b\left[\sin\theta\right]_0^{\frac{\pi}{2}} 2\int_0^{2\pi} \sin\frac{\phi}{2}\, d\left(\frac{\phi}{2}\right)$$

$$= 2b\left[-\cos\frac{\phi}{2}\right]_0^{2\pi} = -2b[-1-1] = 4b$$

$$\int_S \nabla\times\bar{A}\cdot d\bar{s} = \underline{\underline{4b}}$$

$$\oint_C \bar{A}\cdot d\bar{l} \qquad d\bar{l} = \bar{a}_R\, dR + \bar{a}_\theta R\, d\theta + \bar{a}_\phi R\sin\theta\, d\phi$$

$$\bar{A} = \bar{a}_\phi \sin\frac{\phi}{2}$$

$$\oint_C \bar{a}_\phi \sin\frac{\phi}{2} \cdot \left(\cancel{\bar{a}_R\, dR} + \cancel{\bar{a}_\theta R\, d\theta} + \bar{a}_\phi R\sin\theta\, d\phi\right)$$

$$\int_0^{2\pi} \overset{b}{R} \sin\left(\frac{\phi}{2}\right) \overset{1}{\sin\theta}\, d\phi$$

$R = b \rightarrow$ const,
$\theta = \frac{\pi}{2} \rightarrow$ const for Contour C!

$$2b\int_0^{2\pi} \sin\frac{\phi}{2}\, d\left(\frac{\phi}{2}\right) = 2b\left[-\cos\frac{\phi}{2}\right]_0^{2\pi} = 4b$$

$$\oint_C \bar{A}\cdot d\bar{l} = \underline{\underline{4b}}$$ <u>proven</u>