ECSE 353 ELECTROMAGNETIC FIELDS AND WAVES FORMULAS

$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$ $\qquad$ $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$ $\qquad$ $\nabla\times(\psi\mathbf{A}) = \psi\nabla\times\mathbf{A} + \nabla\psi\times\mathbf{A}$

$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$ $\qquad$ $\nabla(\psi V) = \psi\nabla V + V\nabla\psi$

$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ $\qquad$ $\nabla\cdot\nabla V = \nabla^2 V$ $\qquad$ $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ $\qquad$ $\nabla\times\nabla V = 0$

$$\int_V \nabla\cdot\mathbf{A}\,dv = \oint_S \mathbf{A}\cdot d\mathbf{s} \quad \text{(Divergence thm.)} \qquad \int_S \nabla\times\mathbf{A}\cdot d\mathbf{s} = \oint_C \mathbf{A}\cdot d\mathbf{l} \quad \text{(Stokes's thm.)}$$

Cartesian Coordinates (x,y,z) $\qquad$ $\mathbf{dl} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$

$$\nabla V = \mathbf{a}_x\frac{\partial V}{\partial x} + \mathbf{a}_y\frac{\partial V}{\partial y} + \mathbf{a}_z\frac{\partial V}{\partial z} \qquad \nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathrm{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates (r,ϕ,z) $\qquad$ $\mathbf{dl} = dr\,\mathbf{a}_r + r d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z$

$$\nabla V = \mathbf{a}_r\frac{\partial V}{\partial r} + \mathbf{a}_\phi\frac{\partial V}{r\partial\phi} + \mathbf{a}_z\frac{\partial V}{\partial z} \qquad \nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\phi}{r\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{a}_r\left(\frac{\partial A_z}{r\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \mathbf{a}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{r\partial\phi}\right)$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates (R,θ,ϕ) $\qquad$ $\mathbf{dl} = dR\,\mathbf{a}_R + R d\theta\,\mathbf{a}_\theta + R\sin\theta\, d\phi\,\mathbf{a}_\phi$

$$\nabla V = \mathbf{a}_R\frac{\partial V}{\partial R} + \mathbf{a}_\theta\frac{\partial V}{R\partial\theta} + \mathbf{a}_\phi\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathrm{A} = \mathbf{a}_R\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right) + \mathbf{a}_\theta\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial}{\partial R}(R A_\phi)\right)$$
$$+ \mathbf{a}_\phi\frac{1}{R}\left(\frac{\partial}{\partial R}(R A_\theta) - \frac{\partial A_R}{\partial\theta}\right)$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

Electrostatics

$\nabla \cdot \mathbf{D} = \rho \qquad \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$

$\nabla \times \mathbf{E} = 0 \qquad \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \mathbf{E}_{1t} = \mathbf{E}_{2t}$

Potential : $\mathbf{E} = -\nabla V \quad V_{12} = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} \quad \nabla \cdot \varepsilon \nabla V = -\rho$

Sources in Free Space : $\mathbf{E} = \frac{1}{4\pi \varepsilon_0} \mathbf{a}_R \frac{q}{R^2} \qquad V = \frac{1}{4\pi \varepsilon_0} \frac{q}{R}$

Infinite Line Charge : $\mathbf{E} = \mathbf{a}_r \frac{\rho_l}{2\pi \varepsilon_0 r} \qquad V = \frac{\rho_l}{2\pi \varepsilon_0} \ln\left(\frac{r_0}{r}\right)$

Electric Dipole : $\mathbf{p} = q\mathbf{d} \qquad V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi \varepsilon_0 R^2}$

Polarization Charge : $\rho_p = -\nabla \cdot \mathbf{P} \qquad \rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$

Flux Density : $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_r \mathbf{E}$

$C = \frac{Q}{V}$ Parallel Plate: $C = \frac{\varepsilon S}{d}$ Coax: $C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} \; (Fm^{-1})$

Wires: $C = \frac{\pi\varepsilon}{\cosh^{-1}\left(\frac{D}{2a}\right)} \; (Fm^{-1})$

Energy : $W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$ or $\frac{1}{2}\int_{V'} \rho V \, dv'$ or $\frac{1}{2}\int_{V'} \mathbf{E} \cdot \mathbf{D} \, dv'$ or $\frac{1}{2} C V^2$

Physical Constants

$\mu_0 = 4\pi \times 10^{-7}$ (Hm^{-1}) $\qquad \varepsilon_0 \approx \frac{1}{36\pi} \times 10^{-9}$ (Fm^{-1}) $\qquad$ 1 Npm^{-1} = 8.686 dBm^{-1}

$\eta_0 \approx 377\,\Omega$ or $120\pi\,\Omega \qquad c \approx 3 \times 10^8$ (ms^{-1})

Steady Electric Currents

$\mathbf{J} = Nq\mathbf{u} \qquad \int_S \mathbf{J}\cdot d\mathbf{s} = I \qquad \mathbf{J}_s = N_s q\mathbf{u} \qquad \int_C \mathbf{J}_s \cdot \mathbf{a}_n dl = I \qquad \mathbf{J} = \sigma(\mathbf{E}+\mathbf{E}_i)$

$\text{EMF} = \oint_C (\mathbf{E}+\mathbf{E}_i)\cdot d\mathbf{l} \qquad \text{EMF} = I\sum_k R_k \qquad \text{EMF}_k = \int_{\text{Cond } k} \mathbf{E}_i \cdot d\mathbf{l}$

$\nabla\cdot\mathbf{J} = 0 \qquad \oint_S \mathbf{J}\cdot d\mathbf{s} = 0 \qquad J_{1n} = J_{2n}$

Joules' Law: $P = \int_V \mathbf{E}\cdot\mathbf{J}\, dv$ or VI

$R = -\int_2^1 (\mathbf{E}+\mathbf{E}_i)\cdot dl \Big/ I_{12} \qquad R_{cylinder} = d/(\sigma S) \qquad C/G = \varepsilon/\sigma \qquad R_s = 1/(\delta\sigma)$

Magnetostatics and Faraday's Law

$\nabla\cdot\mathbf{B} = 0 \qquad \oint_S \mathbf{B}\cdot d\mathbf{s} = 0 \qquad B_{1n} = B_{2n}$

$\nabla\times\mathbf{H} = \mathbf{J} \qquad \oint_C \mathbf{H}\cdot d\mathbf{l} = I_C \qquad \mathbf{H}_{1t} - \mathbf{H}_{2t} = \mathbf{J}_s \times \mathbf{a}_{n2}$

$\mathbf{m} = \mathbf{a}_z IS \qquad \mathbf{B} = \frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \mathbf{J}_m = \nabla\times\mathbf{M} \qquad \mathbf{J}_{ms} = \mathbf{M}\times\mathbf{a}_n$

Field Intensity: $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} = \frac{1}{\mu_0\mu_r}\mathbf{B}$

Infinite Line Current: $\mathbf{B} = \mathbf{a}_\phi \frac{\mu I}{2\pi r}$ Infinite Solenoid: $\mathbf{B} = \mathbf{a}_z \mu nI$

$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$

$\Lambda_{12} = N_2\Phi_{12} = N_2\int_{S_2} \mathbf{B}_1\cdot d\mathbf{s}_2 \qquad L_{12} = \frac{\Lambda_{12}}{I_1} \qquad L = L_{11} = \frac{\Lambda_{11}}{I_1}$

Coax: $L_{\text{ext}} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$ (Hm^{-1}) Wires: $L_{\text{ext}} = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$ (H m^{-1})

Energy: $W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} I_j I_k$ or $\frac{1}{2}\int_{V'} \mathbf{B}\cdot\mathbf{H}\, dv'$ Wire: $L_{\text{int}} = \frac{\mu}{8\pi}$ (Hm^{-1})

Maxwell's Equations

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \oint_C \mathbf{H}\cdot d\mathbf{l} = I + \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot ds \qquad \mathbf{H}_{1t} - \mathbf{H}_{2t} = \mathbf{J}_s \times \mathbf{a}_{n2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}(t)\cdot d\mathbf{s} \qquad \mathbf{E}_{1t} = \mathbf{E}_{2t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \oint_S \mathbf{B}\cdot d\mathbf{s} = 0 \qquad B_{1n} = B_{2n}$$

$$\nabla \cdot \mathbf{D} = \rho_v \qquad \oint_S \mathbf{D}\cdot d\mathbf{s} = Q \qquad \mathbf{a}_{n2}\cdot(\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

$$\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0 \qquad \mathbf{a}_{n2}\cdot(\mathbf{J}_1 - \mathbf{J}_2) = -\partial\rho_s/\partial t$$

Transmission Lines

$$-\frac{\partial v}{\partial z} = Ri + L\frac{\partial i}{\partial t} \qquad -\frac{\partial i}{\partial z} = Gv + C\frac{\partial v}{\partial t}$$

Sinusoidal Waves on Transmission Lines (Phasors)

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \qquad -\frac{dI(z)}{dz} = (G + j\omega C)V(z) \qquad \frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \qquad \frac{d^2 I(z)}{dz^2} = \gamma^2 I(z)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad u_p = \frac{\omega}{\beta} \qquad \lambda = \frac{2\pi}{\beta}$$

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z} \qquad I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{+\gamma z} \qquad \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Lossless Line: $R = G = 0 \quad \alpha = 0 \quad \beta = \omega\sqrt{LC} \quad Z_0 = \sqrt{\frac{L}{C}} \quad u_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon\mu}}$

Distortionless, $\frac{R}{L} = \frac{G}{C}$: $\alpha = R\sqrt{\frac{C}{L}} \quad \beta = \omega\sqrt{LC} \quad Z_0 = \sqrt{\frac{L}{C}} \quad u_p = \frac{1}{\sqrt{LC}}$

Low-Loss, $R << \omega L, G << \omega C$: $\alpha \approx \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) \quad \beta \approx \omega\sqrt{LC} \quad Z_0 \approx \sqrt{\frac{L}{C}}\left(1 - \frac{j}{2\omega}\left(\frac{R}{L} - \frac{G}{C}\right)\right)$

$$Z_i = \frac{V_i}{I_i} = Z_0 \frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} \qquad \tanh(j\beta l) = j\tan(\beta l)$$

$$V_o^+ = \tfrac{1}{2}(V_i + Z_0 I_i) \qquad V_o^- = \tfrac{1}{2}(V_i - Z_0 I_i) \qquad \Gamma = \frac{V_o^- e^{+\gamma l}}{V_o^+ e^{-\gamma l}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Lossless Line: $|V(z)| = |V_o^+|\,\left|1 + |\Gamma_L|\, e^{j(\theta_\Gamma - 2\beta z')}\right|$ $\quad S = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1+|\Gamma|}{1-|\Gamma|}$

Scattering Parameters

$a = V^+ / \sqrt{Z_0} \quad b = V^- / \sqrt{Z_0} \quad P = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2$

$V = \sqrt{Z_0}(a+b) \qquad I = (a-b)/\sqrt{Z_0} \qquad a = \frac{1}{2}\left(V/\sqrt{Z_0} + I\sqrt{Z_0}\right) \qquad b = \frac{1}{2}\left(V/\sqrt{Z_0} - I\sqrt{Z_0}\right)$

$b_1 = S_{11}a_1 + S_{12}a_2$

$b_2 = S_{21}a_1 + S_{22}a_2$

If $a_1 = 1$ and $a_2 = 0$: $\quad V_1 = (1 + S_{11})\sqrt{Z_{01}} \quad V_2 = S_{21}\sqrt{Z_{02}}$

$I_1 = (1 - S_{11})/\sqrt{Z_{01}} \quad I_2 = -S_{21}/\sqrt{Z_{02}}$

Lossless: $|S_{11}|^2 + |S_{21}|^2 = 1$ and $|S_{12}|^2 + |S_{22}|^2 = 1$

Transients on Lossless Transmission Lines

$$u = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{LC}} \qquad R_0 = \sqrt{\frac{L}{C}} \qquad \Gamma_L = \frac{R_L - R_0}{R_L + R_0} \qquad \Gamma_g = \frac{R_g - R_0}{R_g + R_0}$$

Plane Electromagnetic Waves

$$E_x = E_0 e^{-\gamma z} \qquad H_y = \frac{1}{\eta_c} E_0 e^{-\gamma z} \qquad \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \qquad \gamma = j\omega\sqrt{\varepsilon_c \mu} = \alpha + j\beta \qquad \varepsilon_c = \varepsilon + \frac{\sigma}{j\omega}$$

Phase velocity: $u_p = \frac{\omega}{\beta}$ Wavelength: $\lambda = \frac{2\pi}{\beta}$

For wave in + z direction, if $\frac{E_{0y}}{E_{0x}} = p + jq$: Linearly polarized: $q = 0$

Elliptically polarized: $p = 0, q < 0$ righthand $\quad p = 0, q > 0$ lefthand

Circularly polarized: $p = 0, q = -1$ righthand $\quad p = 0, q = +1$ lefthand

Good conductors: $\alpha \approx \beta \approx \frac{1}{\delta} \qquad \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad \eta_c \approx (1+j)\frac{\alpha}{\sigma} \qquad u_p \approx \sqrt{\frac{2\omega}{\mu\sigma}}$

Poynting Vector: $\wp = \mathbf{E} \times \mathbf{H}$ Poynting's Theorem: $\int_S \wp \cdot d\mathbf{s} = -\frac{\partial W_m}{\partial t} - \frac{\partial W_e}{\partial t} - P_\sigma$

$\wp_{av} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^*\right)$ Plane wave in $+z$ direction: $\wp_{av} = \frac{1}{2\eta}|E_0|^2 \mathbf{a}_z$

Normal Incidence: $\mathbf{E}_i = \mathbf{a}_x E_{io}\, e^{-\gamma_1 z}$ $\quad \mathbf{E}_r = \mathbf{a}_x \Gamma E_{io}\, e^{+\gamma_1 z}$ $\quad \mathbf{E}_t = \mathbf{a}_x \tau E_{io}\, e^{-\gamma_2 z}$

$$\mathbf{H}_i = \mathbf{a}_y \frac{E_{io}}{\eta_1} e^{-\gamma_1 z} \qquad \mathbf{H}_r = -\mathbf{a}_y \Gamma \frac{E_{io}}{\eta_1} e^{+\gamma_1 z} \qquad \mathbf{H}_t = \mathbf{a}_y \tau \frac{E_{io}}{\eta_2} e^{-\gamma_2 z}$$

$\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ $\quad \tau = \dfrac{2\eta_2}{\eta_2 + \eta_1}$ $\quad$ If 2 is perfect conductor: $\Gamma = -1, \ \tau = 0$

Antennas and Radiation

$$V = \frac{q}{4\pi\varepsilon R} e^{-jkR} \qquad \mathbf{A} = \frac{\mu \mathbf{J} dV}{4\pi R} e^{-jkR} = \frac{\mu I d\mathbf{l}}{4\pi R} e^{-jkR}$$

$\mathbf{H} = \dfrac{1}{\mu}\nabla \times \mathbf{A}$ $\quad \mathbf{E} = -\nabla V - j\omega \mathbf{A}$ $\quad$ or $\quad \dfrac{1}{j\omega\varepsilon}\nabla \times \mathbf{H}$

Hertzian dipole: $A_R = \dfrac{\mu I dl}{4\pi}\dfrac{e^{-jkR}}{R}\cos\theta$ $\quad A_\theta = -\dfrac{\mu I dl}{4\pi}\dfrac{e^{-jkR}}{R}\sin\theta$ $\quad A_\phi = 0$

Far-zone fields: $H_\phi = j\dfrac{Idl}{4\pi}\dfrac{e^{-jkR}}{R}k\sin\theta$ $\quad E_\theta = j\dfrac{Idl}{4\pi}\dfrac{e^{-jkR}}{R}\eta k\sin\theta$

Radiation Intensity, $U = \wp_{av} R^2$ $\quad$ Isotropic: $U_{\text{iso}} = \dfrac{P_r}{4\pi}$ $\quad$ Effective area, $A_e = \dfrac{P_L}{\wp_{av}}$

Directive Gain: $G_D = \dfrac{U}{U_{\text{iso}}}$ $\quad$ Directivity: $\dfrac{U_{\max}}{U_{\text{iso}}}$ $\quad$ Gain: $\dfrac{U_{\max}}{(P_i/4\pi)}$ $\quad G_D = \dfrac{4\pi}{\lambda^2}A_e$

Radiation Efficiency: $\eta_r = \dfrac{P_r}{P_i}$