Final exam, 2007

Solutions

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(Version A) 1. D

2. D

3. B

4. B

5. D

6. A

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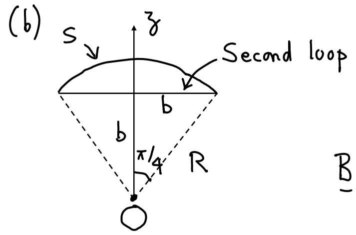
7\. (a) $\underline{m} = \underline{a}_z \, I S = \underline{a}_z \, I \pi a^2$ $\quad (Am^2)$

(b)



$$\Phi = \int_S \underline{B} \cdot d\underline{s}$$

$$\underline{B} = \frac{\mu_0 m}{4\pi R^3} \left(\underline{a}_R \, 2\cos\theta + \underline{a}_\theta \sin\theta \right)$$

$$d\underline{s} = ds \, \underline{a}_R = \left(R \, d\theta \quad R \sin\theta \, d\varphi \right) \underline{a}_R$$

So $$\Phi = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \frac{\mu_0 m}{4\pi R^3} \, 2\cos\theta \quad R^2 \sin\theta \, d\theta \, d\varphi$$

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$$\Phi = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/4} \frac{\mu_0 m}{4\pi R^3} \, 2\cos\theta \quad R^2 \sin\theta \, d\theta \, d\varphi$$

$$= 2\pi \frac{\mu_0 m}{4\pi R} \int_{\theta=0}^{\pi/4} \sin 2\theta \, d\theta = \frac{\mu_0 m}{2R} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/4} = \frac{\mu_0 m}{4R}$$

with $R = \sqrt{b^2 + b^2} = \sqrt{2}\, b$

$m = I \pi a^2$

So

$$\Phi = \frac{\mu_0 I \pi a^2}{4\sqrt{2}\, b} \quad (Wb)$$

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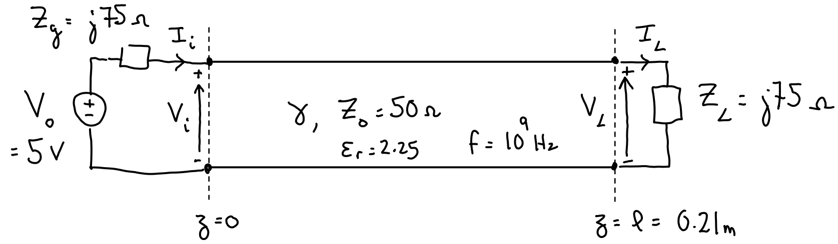

Lossless $\Rightarrow$ $\gamma = j\beta$ $\quad \beta = \omega\sqrt{LC} = \omega\sqrt{\varepsilon\mu} = \dfrac{2\pi f}{c}\sqrt{\varepsilon_r} = \dfrac{2\pi \times 10 \times 3}{3 \times 2} = 10\pi$ rads/m

$\Rightarrow \quad \beta\ell = 10\pi \times 0.21 = 2.1\pi$ rads

$$Z_i = Z_0\left(\frac{Z_L + Z_0\tanh\gamma\ell}{Z_0 + Z_L\tanh\gamma\ell}\right) = 50\left(\frac{j75 + j50\tan 2.1\pi}{50 - 75\tan 2.1\pi}\right)$$

$$= +j178.0\ \Omega$$

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$$V_i = \left(\frac{Z_i}{Z_g + Z_i}\right)V_0 = \left(\frac{+j178.0}{j75 + j178.0}\right)5 = 3.518\ \text{V}$$

$$I_i = \frac{V_0}{Z_g + Z_i} = \frac{5}{j75 + j178.0} = -j\,19.76\ \text{mA}$$

$$V_0^+ = \frac{1}{2}(V_i + Z_0 I_i) = \frac{1}{2}\left(3.518 - j50 \times 19.76 \times 10^{-3}\right) = 1.759 - j0.494$$

$$V_0^- = \frac{1}{2}(V_i - Z_0 I_i) = \frac{1}{2}\left(3.518 + j50 \times 19.76 \times 10^{-3}\right) = 1.759 + j0.494$$

$$V_L = V_0^+ e^{-j\beta\ell} + V_0^- e^{+j\beta\ell} = V_0^+ e^{-j\beta\ell} + \left(V_0^+ e^{-j\beta\ell}\right)^*$$

$$= 2\,\mathrm{Re}\, V_0^+ e^{-j\beta\ell} = 2 \times \left(1.759\cos\beta\ell - 0.494\sin\beta\ell\right)$$

$$= 3.04\ \text{V}$$

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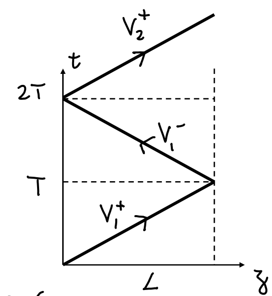
9. (a) $V_1^+ = V_0 \dfrac{R_0}{R_0+R_g} \Rightarrow R_0(V_0 - V_1^+) = R_g V_1^+$

$$\Rightarrow R_0 = R_g \frac{V_1^+}{V_0 - V_1^+} = 100\left(\frac{1}{3-1}\right) = 50\ \Omega$$

(b) $T = \dfrac{L}{u_p} \Rightarrow L = u_p T$

But $u_p = c$ (air filled)

and $2T = 4$ ns



So $L = 3\times10^8 \times \dfrac{4}{2} \times 10^{-9} = 0.6$ m

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(c) Just after 4 ns: $V_i = V_1^+ + V_1^- + V_2^+$

$$= V_1^+ + \Gamma_L V_1^+ + \Gamma_L \Gamma_g V_1^+$$

$$\Rightarrow \Gamma_L(1+\Gamma_g) = \frac{V_i}{V_1^+} - 1 \Rightarrow \Gamma_L = \frac{1}{(1+\Gamma_g)}\left(\frac{V_i}{V_1^+} - 1\right)$$

We have: $\Gamma_g = \dfrac{R_g - R_0}{R_g + R_0} = \dfrac{100-50}{100+50} = \dfrac{1}{3}$, $V_1^+ = 1$, $V_i = 1.2$

So $\Gamma_L = \dfrac{1}{(1+\frac{1}{3})}(1.2-1) = \dfrac{3}{4}\times\dfrac{1}{5} = \dfrac{3}{20}$ But $\Gamma_L = \dfrac{R_L - R_0}{R_L + R_0}$

$$\Rightarrow R_L(1-\Gamma_L) = R_0(1+\Gamma_L)$$

$$\Rightarrow R_L = R_0\left(\frac{1+\Gamma_L}{1-\Gamma_L}\right) = 50\left(\frac{1+3/20}{1-3/20}\right) = 50\left(\frac{23}{17}\right) = \underline{67.6\ \Omega}$$

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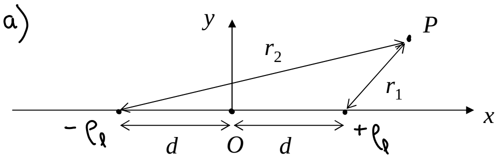
10. (a)



Due to $+\rho_\ell$: $\quad V_{P0} = \dfrac{\rho_\ell}{2\pi\varepsilon_0} \ln \dfrac{d}{r_1}$

Due to $-\rho_\ell$: $\quad V_{P0} = \dfrac{-\rho_\ell}{2\pi\varepsilon_0} \ln \dfrac{d}{r_2}$

Due to both: $\quad V_{P0} = \dfrac{\rho_\ell}{2\pi\varepsilon_0} \left[\ln \dfrac{d}{r_1} + \ln \dfrac{r_2}{d} \right] = \dfrac{\rho_\ell}{2\pi\varepsilon_0} \ln \dfrac{r_2}{r_1}$

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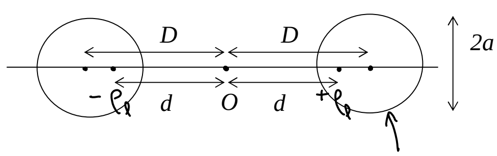
(b)



Centre $(x, y) = (D, 0) = (d\beta, 0)$

Radius $a = d\sqrt{\beta^2 - 1}$

So $\quad d\beta = D$

$d\sqrt{\beta^2-1} = a$

Solving: $\quad d^2\left(\dfrac{D^2}{d^2} - 1\right) = a^2 \quad \Rightarrow \quad d^2 = D^2 - a^2$

$d = \sqrt{D^2 - a^2}$

This makes the right circle an equipotential.

By symmetry, the left circle will also be an equipotential.

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11 (a) $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad \rightarrow \quad \nabla \times \underline{E} = -j\omega\mu_0 \underline{H}$ phasors, free space

$$= -j k_0 \eta_0 \underline{H}$$

From FS: $\nabla \times \mathrm{A} = \mathbf{a}_R \frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right) + \mathbf{a}_\theta \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial}{\partial R}(R A_\phi)\right) + \mathbf{a}_\phi \frac{1}{R}\left(\frac{\partial}{\partial R}(R A_\theta) - \frac{\partial A_R}{\partial\theta}\right)$

So $\nabla \times \underline{E} = \underline{a}_R \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(E_\varphi \sin\theta\right) - \underline{a}_\theta \frac{1}{R}\frac{\partial}{\partial R}\left(R E_\varphi\right)$

$$= \underline{a}_R \frac{1}{R\tan\theta} E_\varphi - \underline{a}_\theta (-jk_0) E_\varphi$$

and

$$\underline{H} = \frac{1}{-jk_0\eta_0}\left(\underline{a}_R \frac{1}{R\tan\theta} + \underline{a}_\theta jk_0\right) \cos\varphi \frac{e^{-jk_0R}}{R}$$

$$= \left(\frac{-1}{jk_0R\tan\theta}\underline{a}_R - \underline{a}_\theta\right)\frac{\cos\varphi}{\eta_0}\frac{e^{-jk_0R}}{R}$$

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(b) In far zone, only $\frac{1}{R}$ fields are considered

So $\underline{H} = -\underline{a}_\theta \frac{\cos\varphi}{\eta_0}\frac{e^{-jk_0R}}{R}$

(c) $\underline{\mathcal{P}}_{av} = \frac{1}{2}\mathcal{R}e\, \underline{E} \times \underline{H}^* = \frac{1}{2}\mathcal{R}e\left(\underline{a}_\varphi \frac{\cos\varphi}{R} e^{-jk_0R} \times (-\underline{a}_\theta)\frac{\cos\varphi}{\eta_0 R} e^{+jk_0R}\right)$

$$= \frac{1}{2}\frac{\cos^2\varphi}{\eta_0 R^2}\underline{a}_R$$

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(d) $P_r = \int_{\text{Sphere}} \underline{\wp}_{av} \cdot \underline{a}_R \, ds = \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{2} \frac{\cos^2\varphi}{\eta_0 R^2} R d\theta \, R\sin\theta \, d\varphi$

$$= \frac{1}{2\eta_0} \Big[-\cos\theta \Big]_0^{\pi} \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right]_0^{2\pi}$$

$$= \frac{1}{2\eta_0} \times 2 \times \pi = \pi / \eta_0$$

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