Final exam, 2006

Solutions

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(Version A) 1. C ($E_R = H_R = 0$ in far zone, so phase is not defined)

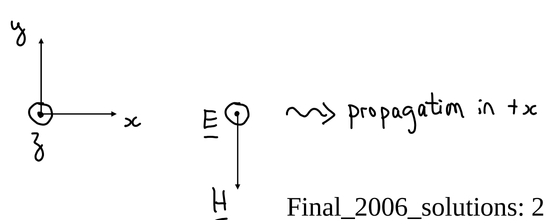
2. C

3. A

4. B Displacement current $= \frac{\partial D}{\partial t}$, but $\frac{\partial}{\partial t} = 0$

5. B

6. B



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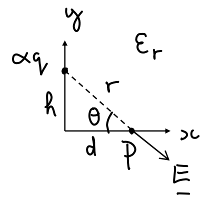
7.

(a)



$$E = \frac{\alpha q}{4\pi\varepsilon_0\varepsilon_r r^2}$$

$$E_x = E\cos\theta = \frac{\alpha q}{4\pi\varepsilon_0\varepsilon_r r^2}\cdot\frac{d}{r}$$

$$E_y = -E\sin\theta = -\frac{\alpha q}{4\pi\varepsilon_0\varepsilon_r r^2}\frac{h}{r}$$

$$\underline{E} = \frac{\alpha q}{4\pi\varepsilon_0\varepsilon_r r^3}\left(d\,\underline{a}_x - h\,\underline{a}_y\right)$$

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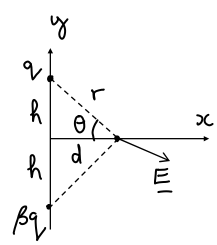
(b)



Due to q:

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^3}\left(d\,\underline{a}_x - h\,\underline{a}_y\right)$$

Due to βq:

$$\underline{E} = \frac{\beta q}{4\pi\varepsilon_0 r^3}\left(d\,\underline{a}_x + h\,\underline{a}_y\right)$$

Total:

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 r^3}\left[d(1+\beta)\,\underline{a}_x - h(1-\beta)\,\underline{a}_y\right]$$

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(c) CASE A: $\underline{E} = \dfrac{\alpha q}{4\pi\varepsilon_0\varepsilon_r r^3}\left(d\,\underline{a}_x - h\,\underline{a}_y \right)$

CASE B: $\underline{E} = \dfrac{q}{4\pi\varepsilon_0 r^3}\left[d(1+\beta)\,\underline{a}_x - h(1-\beta)\,\underline{a}_y \right]$

$$\left.\begin{aligned} E_x \text{ matches: } \quad & \frac{\alpha}{\varepsilon_r} = (1+\beta) \\ D_y \text{ matches: } \quad & \alpha = (1-\beta) \end{aligned}\right\} \quad \begin{aligned} (1-\beta) &= \varepsilon_r(1+\beta) \\ \beta &= \frac{1-\varepsilon_r}{1+\varepsilon_r} \\ \alpha &= \frac{2\varepsilon_r}{1+\varepsilon_r} \end{aligned}$$

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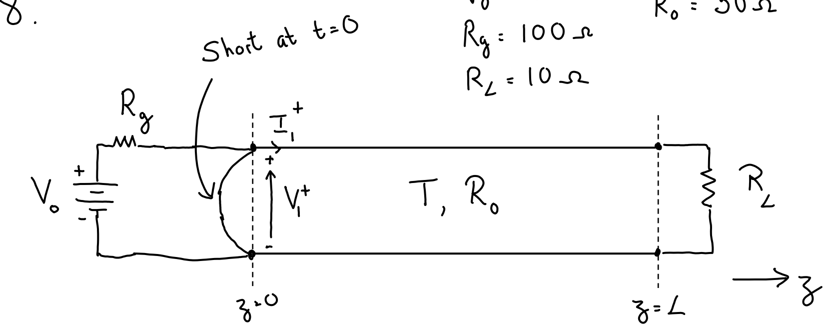
8.

$V_0 = 10$ V

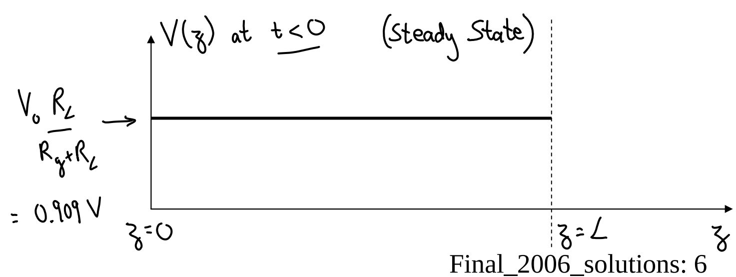
$R_g = 100\,\Omega$

$R_L = 10\,\Omega$

$R_0 = 50\,\Omega$



(a)



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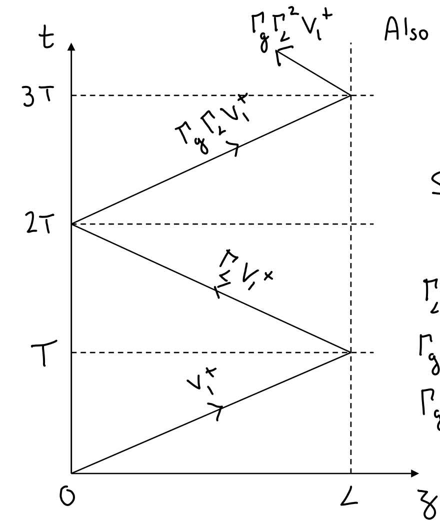
(b) For $t < 0$, $V(0) = 0.909\text{ V}$
For $t > 0$, $V(0) = 0$

So at $t = 0$ this is launched:

$$V_1^+ = -0.909\text{ V}$$

(c)



Also $\Gamma_L = \dfrac{R_L - R_0}{R_L + R_0} = \dfrac{10 - 50}{10 + 50} = -\dfrac{2}{3}$

$\Gamma_g = -1$ (because of short circuit)

So

$V_1^+ = -0.909\text{ V}$

$\Gamma_L V_1^+ = +0.606\text{ V}$

$\Gamma_g \Gamma_L V_1^+ = -0.606\text{ V}$

$\Gamma_g \Gamma_L^2 V_1^+ = 0.404\text{ V}$

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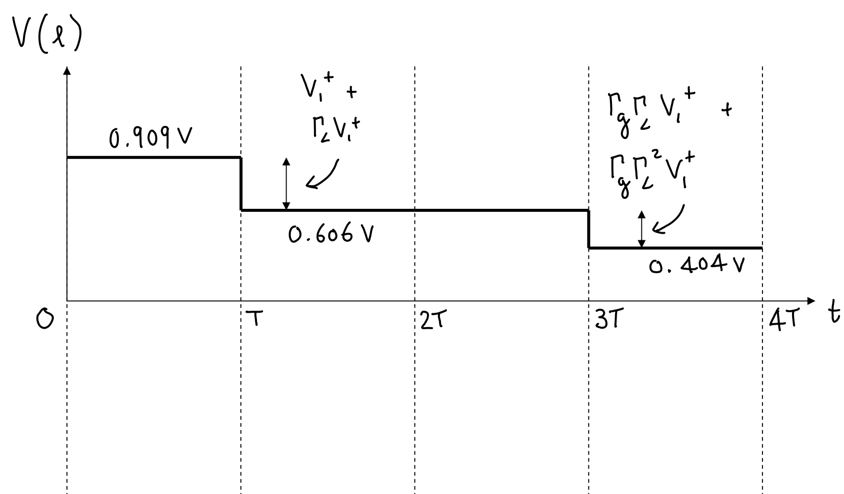

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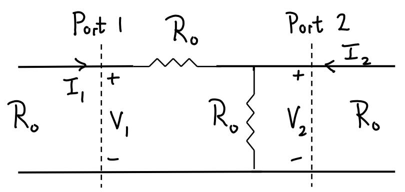
9.



① $a_1 = 1 \quad a_2 = 0$

From Formula Sheet:

$$V_1 = \sqrt{R_o}\,(1 + S_{11}) \qquad V_2 = \sqrt{R_o}\,S_{21}$$

$$I_1 = \frac{1}{\sqrt{R_o}}\,(1 - S_{11}) \qquad I_2 = -\frac{1}{\sqrt{R_o}}\,S_{21}$$

②

KVL: $V_1 = V_2 + I_1 R_0 \Rightarrow \sqrt{R_0}\,(1+S_{11}) = \sqrt{R_o}\,S_{21} + \frac{1}{\sqrt{R_o}}(1-S_{11})\,R_o$

KCL: $I_1 + I_2 = \frac{V_2}{R_0} \Rightarrow \frac{1}{\sqrt{R_0}}(1-S_{11}) - \frac{1}{\sqrt{R_0}}S_{21} = \frac{\sqrt{R_0}}{R_0}S_{21}$

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③ Solve
$$\begin{cases} (1+S_{11}) = S_{21} + (1-S_{11}) \\ (1-S_{11}) - S_{21} = S_{21} \end{cases}$$

$$\Rightarrow \quad 2S_{11} - S_{21} = 0$$
$$S_{11} + 2S_{21} = 1$$

$$\Rightarrow \quad 5S_{11} = 1 \qquad S_{11} = 1/5 \qquad S_{21} = 2/5$$

We know that $S_{12} = S_{21} = 2/5$, but $S_{22} \neq S_{11}$.

So we need to set $a_1 = 0$, $a_2 = 1$.

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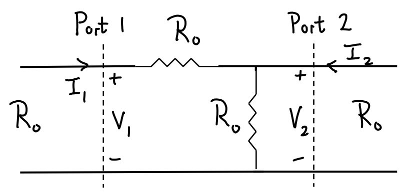

④ $a_1 = 0 \quad a_2 = 1$

$$V_2 = \sqrt{R_0}\,(1+S_{22}) \qquad V_1 = \sqrt{R_0}\,S_{12}$$

$$I_2 = \frac{1}{\sqrt{R_0}}\,(1-S_{22}) \qquad I_1 = \frac{-1}{\sqrt{R_0}}\,S_{12}$$

⑤

KVL: $V_1 = V_2 + I_1 R_0 \Rightarrow \sqrt{R_0}\,S_{12} = \sqrt{R_0}\,(1+S_{22}) - \frac{R_0}{\sqrt{R_0}}\,S_{12}$

KCL: $I_1 + I_2 = \frac{V_2}{R_0} \Rightarrow -\frac{1}{\sqrt{R_0}}\,S_{12} + \frac{1}{\sqrt{R_0}}\,(1-S_{22}) = \frac{\sqrt{R_0}}{R_0}\,(1+S_{22})$

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⑥

$$S_{12} = (1+S_{22}) - S_{12}$$

$$-S_{12} + (1-S_{22}) = (1+S_{22})$$

$$\Rightarrow \quad S_{22} - 2S_{12} = -1$$

$$2S_{22} + S_{12} = 0$$

$$\Rightarrow \quad 5S_{22} = -1 \Rightarrow S_{22} = -\tfrac{1}{5} \qquad S_{12} = 2/5 \quad (= S_{21})$$

So $[S] = \begin{bmatrix} +\frac{1}{5} & +2/5 \\ +2/5 & -1/5 \end{bmatrix}$

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10. $\underline{J}_s = J_{s\varphi}\,\underline{a}_\varphi + J_{sz}\,\underline{a}_z$

Consider each term separately :

1 $J_{s\varphi}\,\underline{a}_\varphi$

This is equivalent to a solenoid with n turns/m of fine wire carrying current I, where

$$nI = J_{s\varphi}$$

From Formula Sheet $\underline{B} = \mu_0 n I\,\underline{a}_z = \mu_0 J_{s\varphi}\,\underline{a}_z$ inside

Also $\underline{B} = 0$ outside

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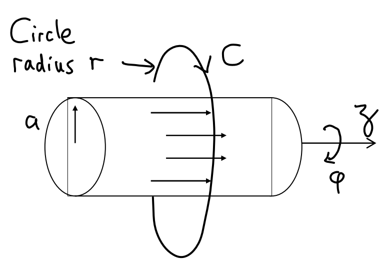
2 $J_{sz}\,\underline{a}_z$ For $r > a$: $\oint_C \underline{H}\cdot d\underline{l} = I_c$



$$H_\varphi\, 2\pi r = J_{sz}\, 2\pi a$$

$$\Rightarrow H_\varphi = J_{sz}\, a/r$$

and $B_\varphi = \mu_0 J_{sz}\, a/r$

For $r < a$: $H_\varphi\, 2\pi r = 0$

$$\Rightarrow B_\varphi = 0$$

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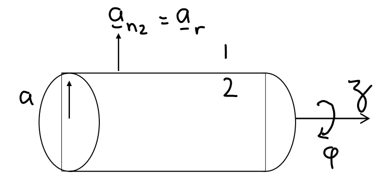
Now add the two parts together:

(a) Inside $\quad \underline{B}_{in} = \mu_0 J_{s\varphi}\, \underline{a}_z + 0 = \mu_0 J_{s\varphi}\, \underline{a}_z$

(b) Outside $\quad \underline{B}_{out} = 0 + \mu_0 J_{sz} \frac{a}{r}\, \underline{a}_\varphi = \mu_0 J_{sz} \frac{a}{r}\, \underline{a}_\varphi$

(c)

$$B_{1n} = B_{out} \cdot \underline{a}_r = 0 = \underline{B}_{in} \cdot \underline{a}_r = B_{2n} \quad \checkmark$$



$$\underline{H}_{1t} = \underline{H}_{out} = \underline{B}_{out}/\mu_0 = J_{sz}\, \underline{a}_\varphi$$

$$\underline{H}_{2t} = \underline{H}_{in} = \underline{B}_{in}/\mu_0 = J_{s\varphi}\, \underline{a}_z$$

So

$$\underline{J}_s \times \underline{a}_{n2} = (J_{s\varphi}\, \underline{a}_\varphi + J_{sz}\, \underline{a}_z) \times \underline{a}_r = J_{sz}\, \underline{a}_\varphi - J_{s\varphi}\, \underline{a}_z = \underline{H}_{1t} - \underline{H}_{t2} \quad \checkmark$$

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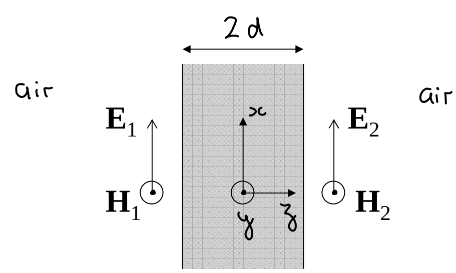
11.



(a) $\quad \underline{E} = \underline{a}_x \left(E^+ e^{-jkz} + E^- e^{+jkz} \right)$

$\quad \underline{H} = \underline{a}_y \left(\frac{E^+}{\eta} e^{-jkz} - \frac{E^-}{\eta} e^{+jkz} \right)$

(b) $\quad \underline{E}$ & $\underline{H}$ are tangentially continuous at $z = \pm d$

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So:

At $z=-d$: $E_1 = E^+ e^{+jkd} + E^- e^{-jkd}$

At $z=+d$: $E_2 = E^+ e^{-jkd} + E^- e^{+jkd}$

At $z=-d$: $\eta H_1 = E^+ e^{+jkd} - E^- e^{-jkd}$

At $z=+d$: $\eta H_2 = E^+ e^{-jkd} - E^- e^{+jkd}$

$$E_1 = E^+ e^{+jkd} + E^- e^{-jkd}$$
$$\eta H_1 = E^+ e^{+jkd} - E^- e^{-jkd}$$

$$2E^+ e^{+jkd} = (E_1 + \eta H_1) \Rightarrow E^+ = \frac{1}{2}(E_1 + \eta H_1)e^{-jkd}$$
$$2E^- e^{-jkd} = (E_1 - \eta H_1) \Rightarrow E^- = \frac{1}{2}(E_1 - \eta H_1)e^{+jkd}$$

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$$E_2 = E^+ e^{-jkd} + E^- e^{+jkd}$$
$$\eta H_2 = E^+ e^{-jkd} - E^- e^{+jkd}$$

$$\Rightarrow E_2 = \frac{1}{2}(E_1 + \eta H_1)e^{-2jkd} + \frac{1}{2}(E_1 - \eta H_1)e^{+2jkd}$$
$$= E_1 \cos 2kd - j\eta H_1 \sin 2kd$$
$$\eta H_2 = \frac{1}{2}(E_1 + \eta H_1)e^{-2jkd} - \frac{1}{2}(E_1 - \eta H_1)e^{+2jkd}$$
$$= -jE_1 \sin 2kd + \eta H_1 \cos 2kd$$

So:
$$E_2 = E_1 \cos 2kd - j\eta H_1 \sin 2kd$$
$$\eta H_2 = -jE_1 \sin 2kd + \eta H_1 \cos 2kd$$

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