ECSE 210: Circuit Analysis

Lecture #9: Second Order Circuits

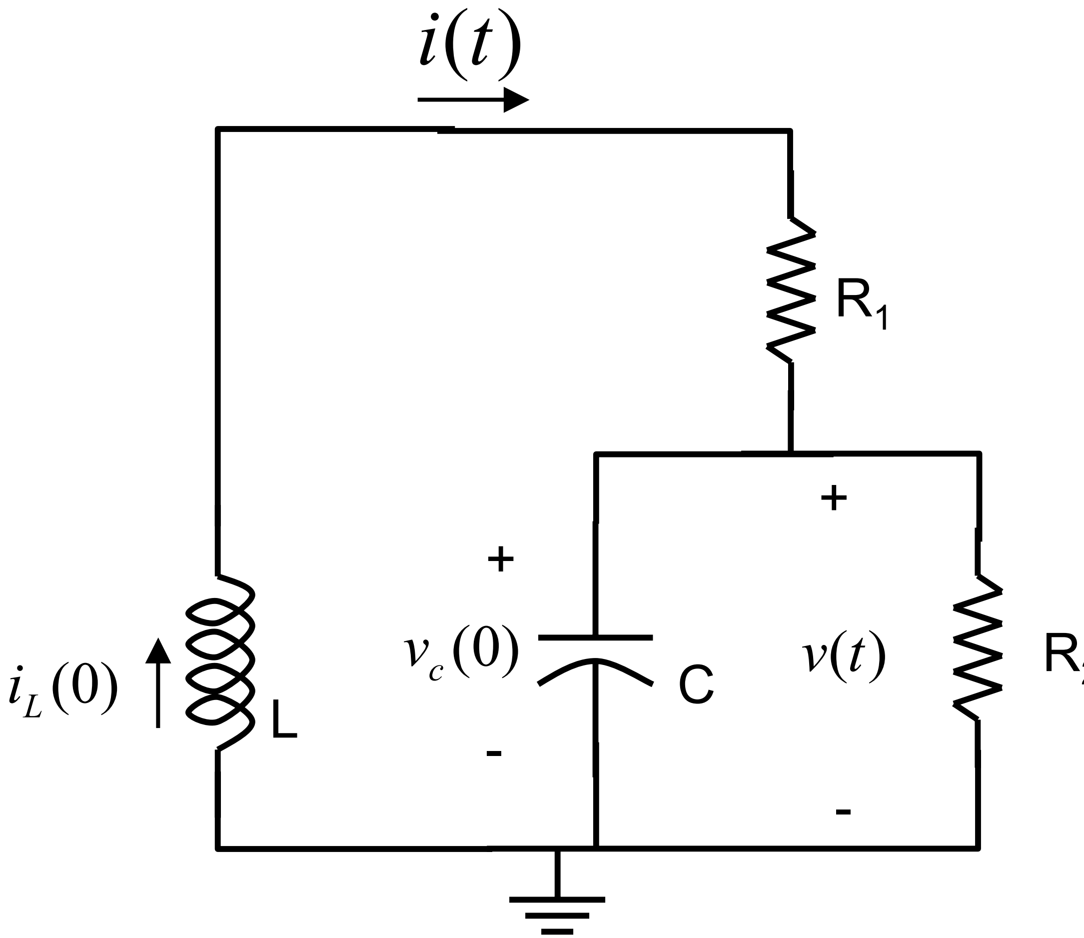
Example 3: Critically Damped

C=0.125F $R_1=10\Omega$

L=2H $R_2=8\Omega$

$i_L(0) = 0.5A$

$v_c(0) = 1V$



KVL $$L\frac{di}{dt} + R_1 i + v = 0$$

KCL $$i(t) = C\frac{dv}{dt} + \frac{v}{R_2}$$

Combine two first order ODE's into one second-order ODE in *v(t)* and solve.

Example 3: Critically Damped

KVL $$L\frac{di}{dt}+R_1 i+v=0$$

KCL $$i(t)=C\frac{dv}{dt}+\frac{v}{R_2}$$

$$\frac{di}{dt}=C\frac{d^2v}{dt^2}+\frac{1}{R_2}\frac{dv}{dt}$$

KVL $$L\left(C\frac{d^2v}{dt^2}+\frac{1}{R_2}\frac{dv}{dt}\right)+R_1\left(C\frac{dv}{dt}+\frac{v}{R_2}\right)+v=0$$

$$LC\frac{d^2v}{dt^2}+\left(\frac{L}{R_2}+R_1C\right)\frac{dv}{dt}+\left(R_1+\frac{1}{R_2}\right)v=0$$

$$\frac{d^2v}{dt^2}+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\frac{dv}{dt}+\left(\frac{R_1+R_2}{R_2LC}\right)v=0$$

Example 3: Critically Damped

$$\frac{d^2v}{dt^2}+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\frac{dv}{dt}+\left(\frac{R_1+R_2}{R_2LC}\right)v=0$$

Recall

$$\frac{d^2x(t)}{dt^2}+2\alpha\frac{dx(t)}{dt}+\omega_o^2x(t)=0$$

$$\frac{d^2v}{dt^2}+6\frac{dv}{dt}+9v=0$$

Damping coefficient $\alpha=3$

Resonant Frequency $\omega_o=3$

→ Critically damped response

$$s^2+6s+9=0$$

→ Roots λ_1=-3 and λ_2=-3. The roots are *real* and *equal*

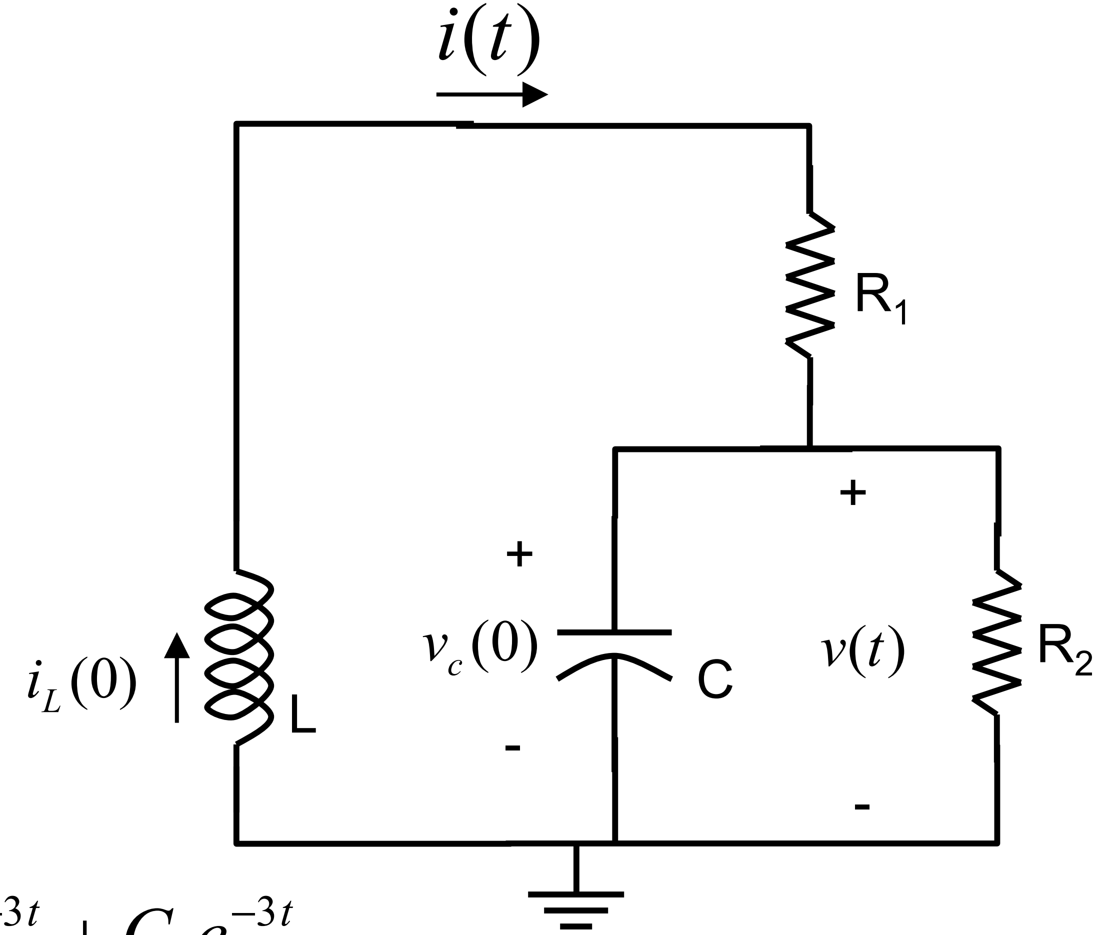
Example 3: Critically Damped

C=0.125F R_1=10Ω

L=2H R_2=8Ω

$i_L(0) = 0.5A$

$v_c(0) = 1V$



$$v(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

$$v(0) = C_1 = 1V$$

$$\frac{dv(t)}{dt} = -3C_1 e^{-3t} - 3C_2 t e^{-3t} + C_2 e^{-3t}$$

$$\frac{dv(0)}{dt} = -3C_1 + C_2$$

Example 3: Critically Damped

C=0.125F R_1=10Ω $i_L(0) = 0.5A$

L=2H R_2=8Ω $v_c(0) = 1V$

$$\frac{dv(0)}{dt} = -3C_1 + C_2$$

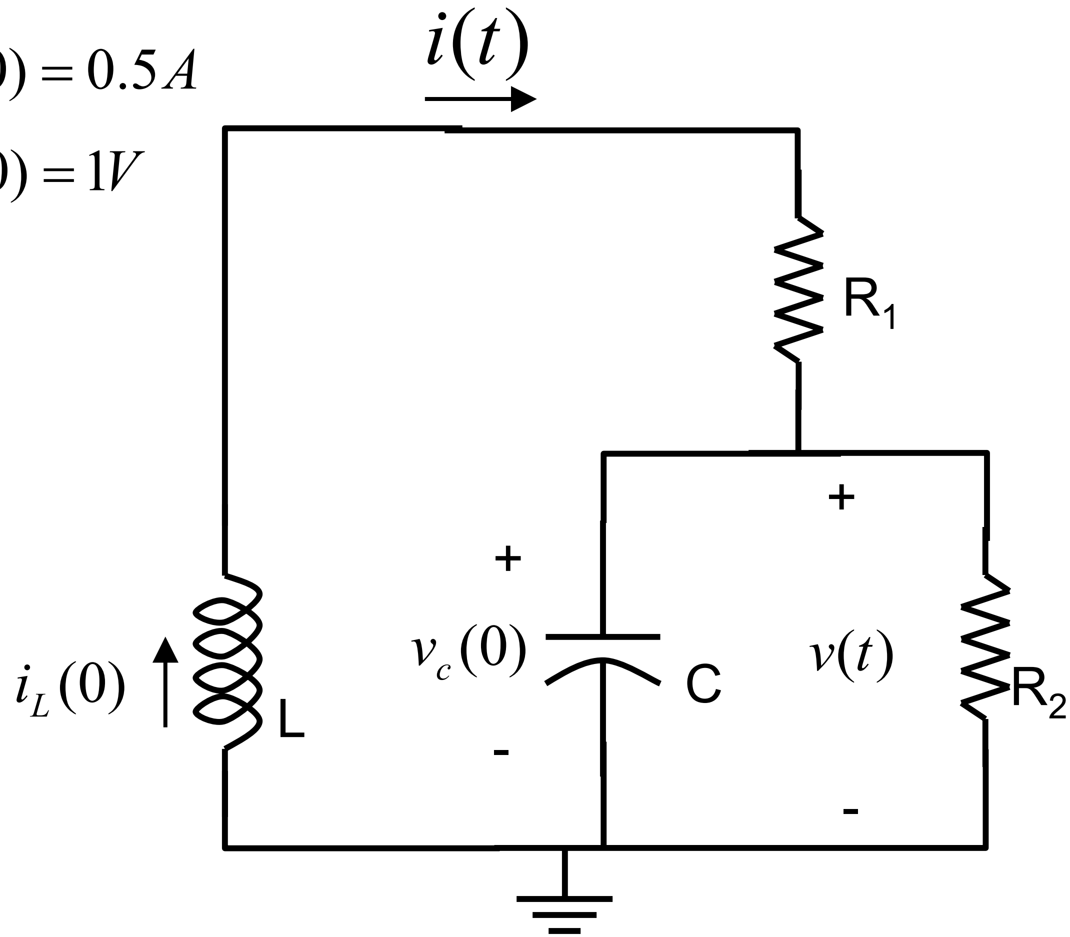
KCL $$C\frac{dv}{dt} + \frac{v}{R_2} - i = 0$$

$$\frac{dv(0)}{dt} = -\frac{v(0)}{CR_2} + \frac{i(0)}{C} = -1 + 4 = 3$$

$$-3C_1 + C_2 = 3$$

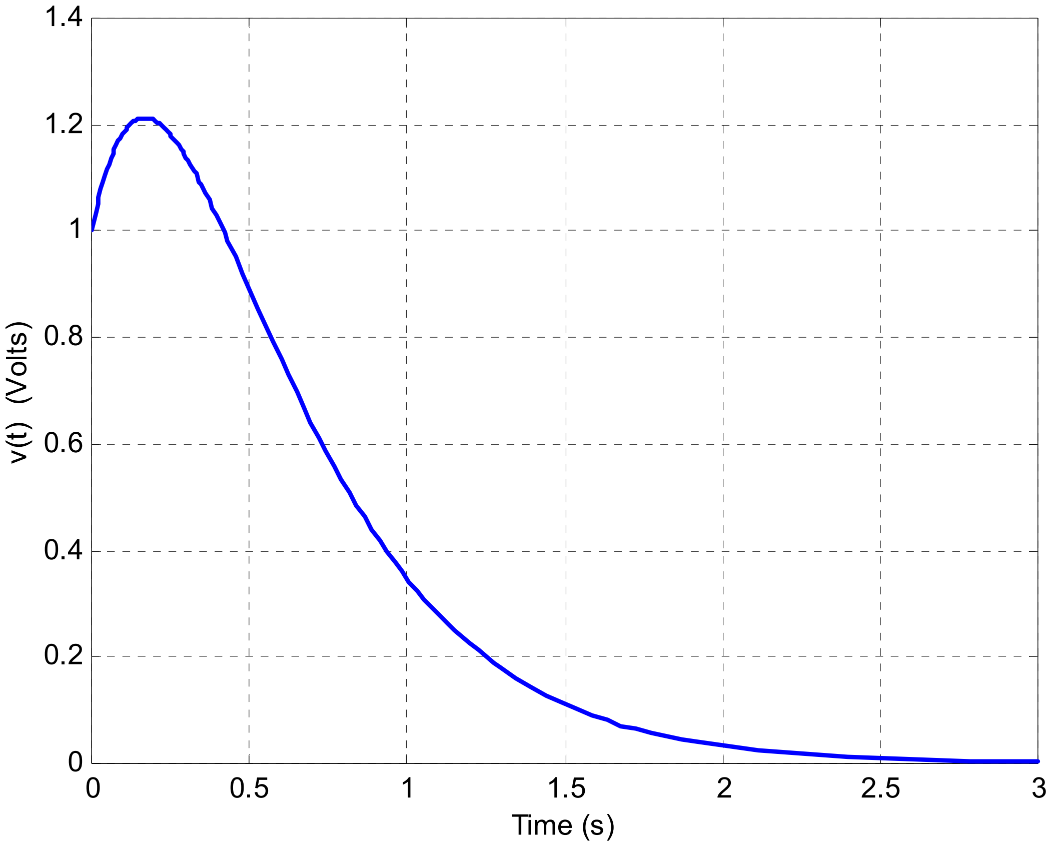
$$C_1 = 1 \quad C_2 = 6$$

$$\rightarrow \quad v(t) = e^{-3t} + 6te^{-3t} \quad \text{V}$$



Example 3: Critically Damped

Volts



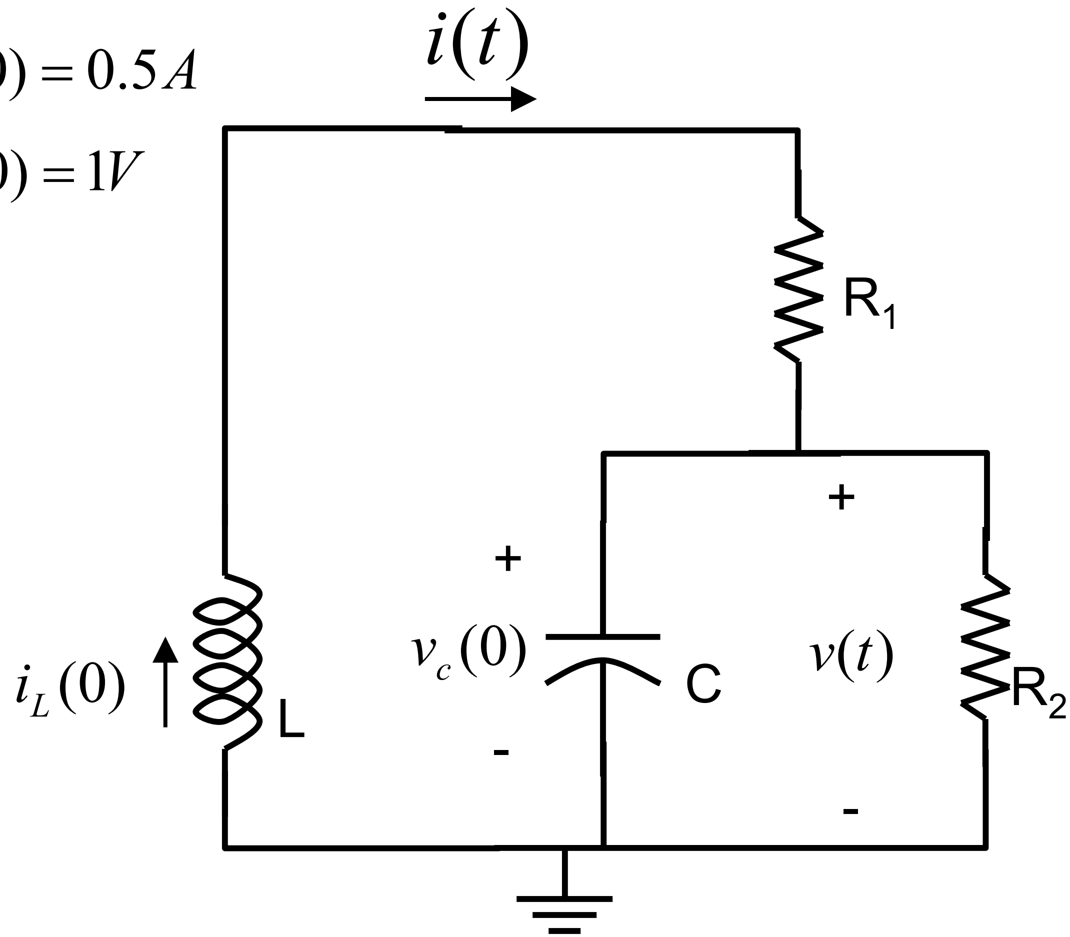
Seconds

$$\rightarrow \quad v(t) = e^{-3t} + 6te^{-3t}$$

Example 3: Critically Damped

C=0.125F $R_1=10\Omega$ $i_L(0)=0.5A$

L=2H $R_2=8\Omega$ $v_c(0)=1V$



$i(t)=?$

$$i(t)=C_1e^{-3t}+C_2te^{-3t}$$

$$i(0)=C_1=0.5$$

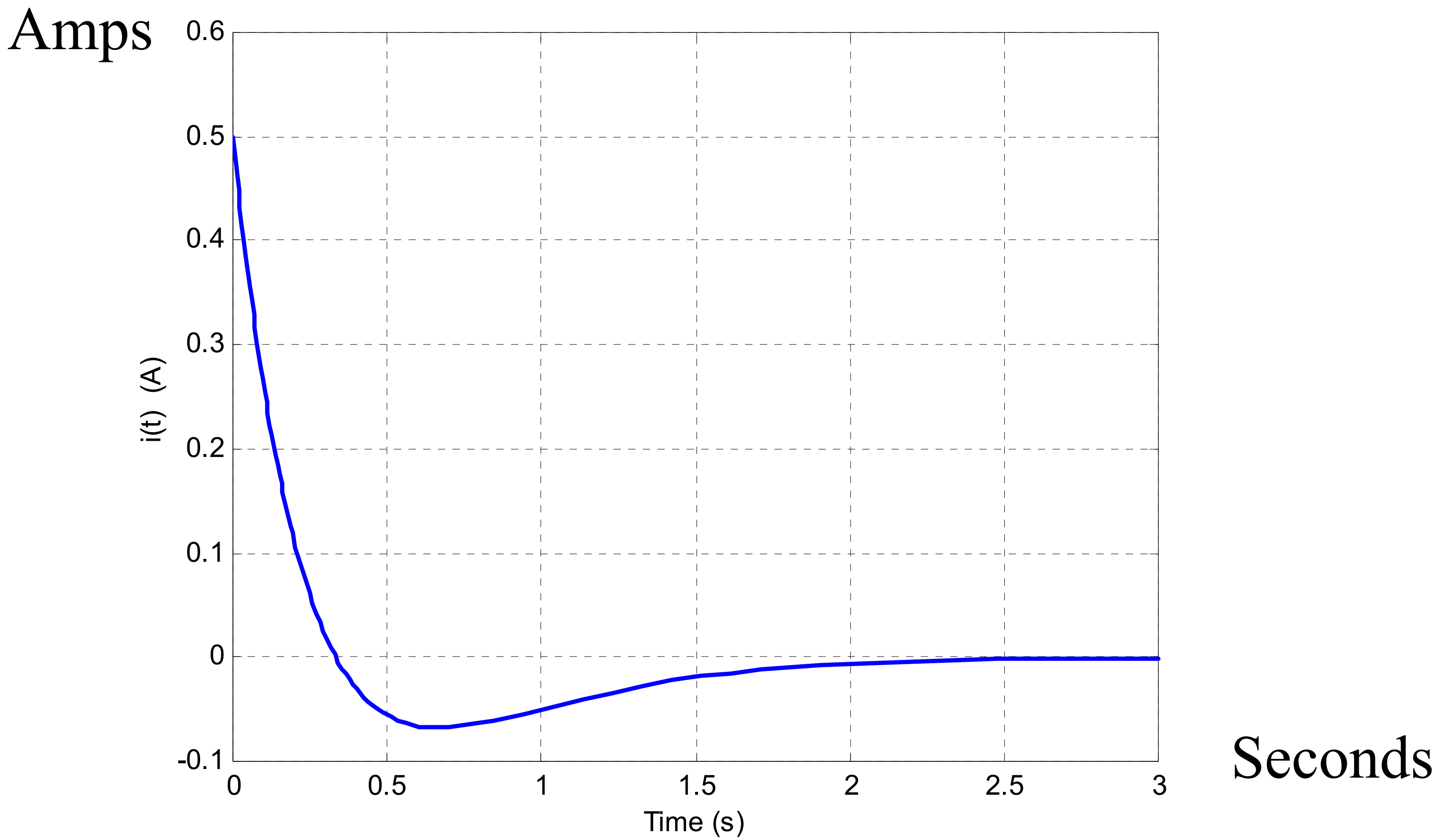
$$\frac{di(0)}{dt}=-3C_1+C_2$$

KVL $\quad L\frac{di}{dt}+R_1i+v=0$

$$\frac{di(0)}{dt}=-\frac{R_1}{L}i(0)-\frac{v(0)}{L}=-3$$

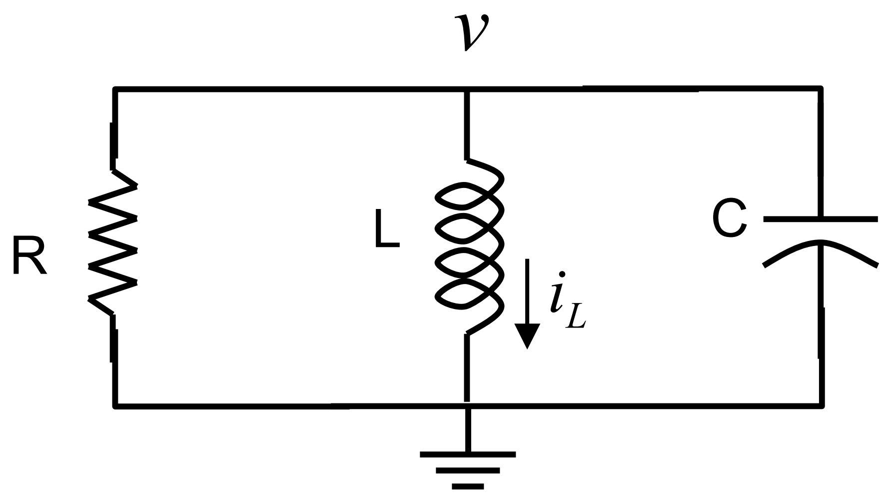
$$\left.\begin{aligned}C_1&=0.5\\ -3C_1+C_2&=-3\end{aligned}\right\}\ C_2=-1.5$$

Example 3: Critically Damped



$$\rightarrow \quad i(t) = 0.5e^{-3t} - 1.5te^{-3t} \qquad t > 0$$

Back to Example 1: Overdamped



R=2Ω

L=5H

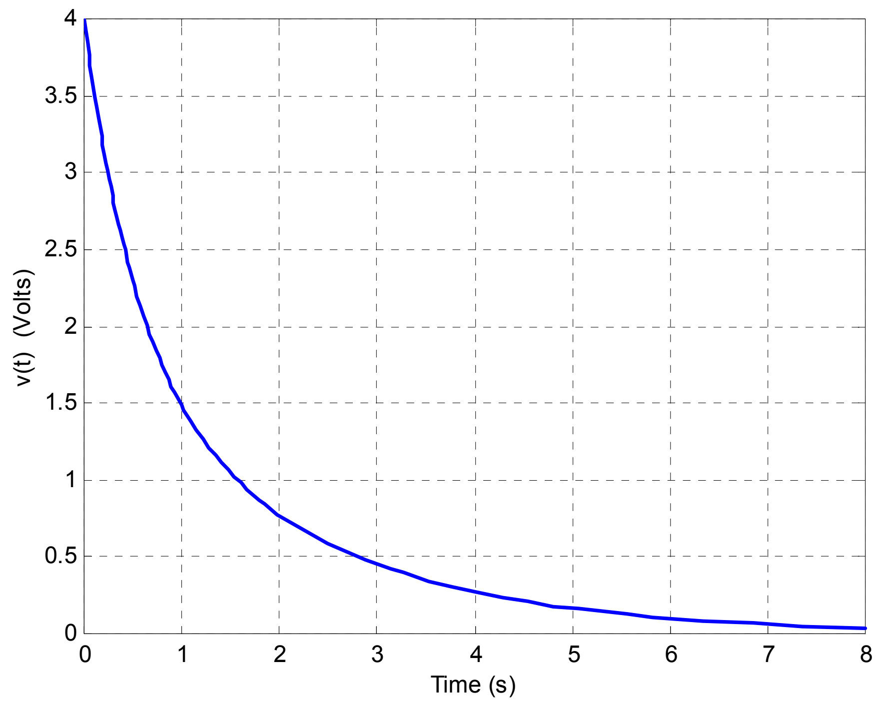
C=0.2F

$$i_L(0) = -1A$$

$$v(0) = 4V$$

$$\rightarrow \quad v(t) = 2e^{-2t} + 2e^{-0.5t} \quad \text{V}$$

Volts



Seconds

Back to Example 2: Underdamped

R=6Ω

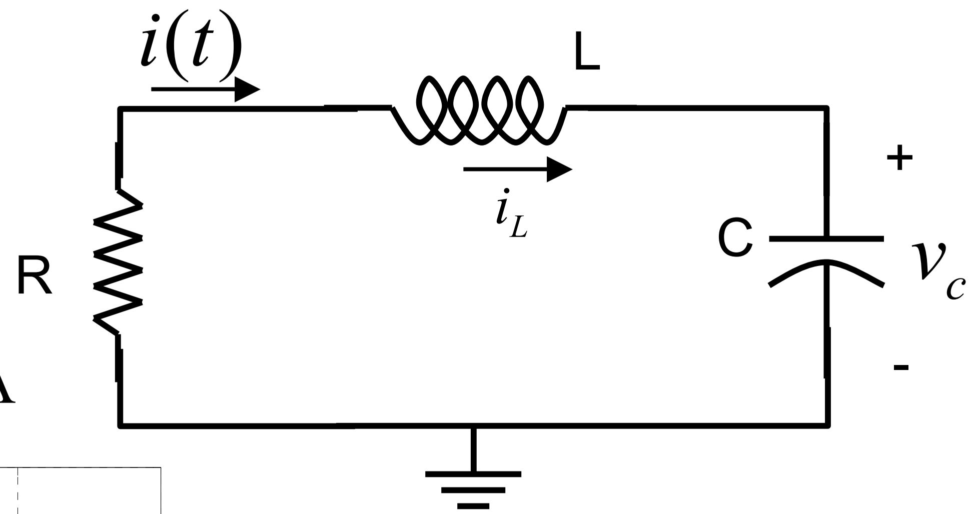
L=1H

C=0.04F

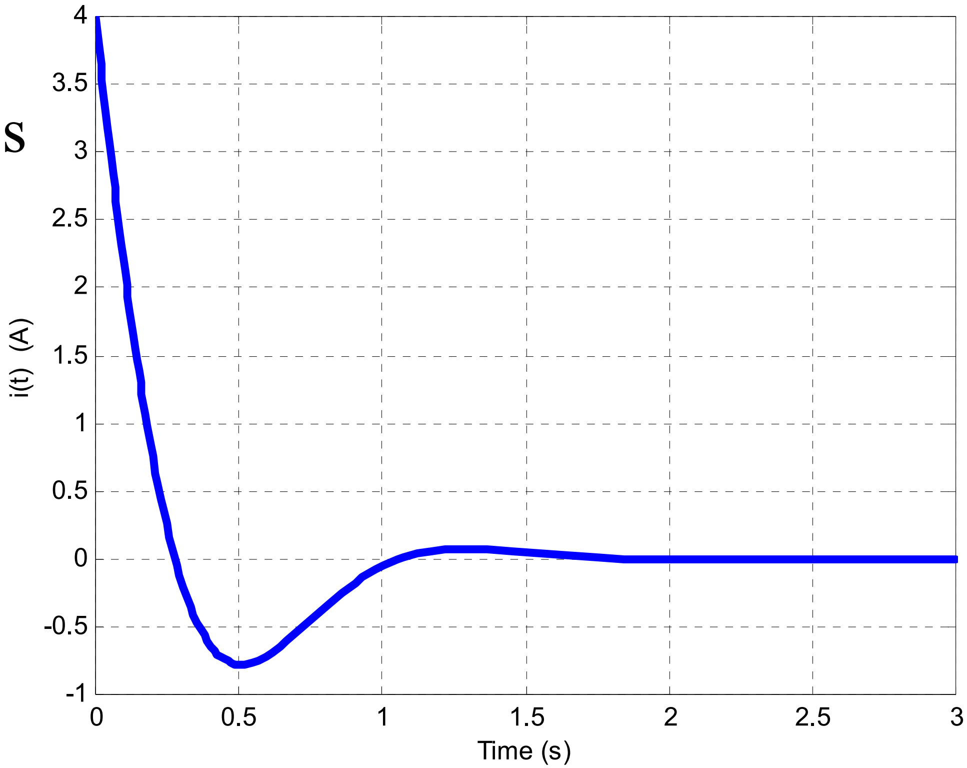
$$i_L(0) = 4A$$

$$v_c(0) = -4V$$

$$i(t) = 4e^{-3t}\cos(4t) - 2e^{-3t}\sin(4t) \text{ A}$$

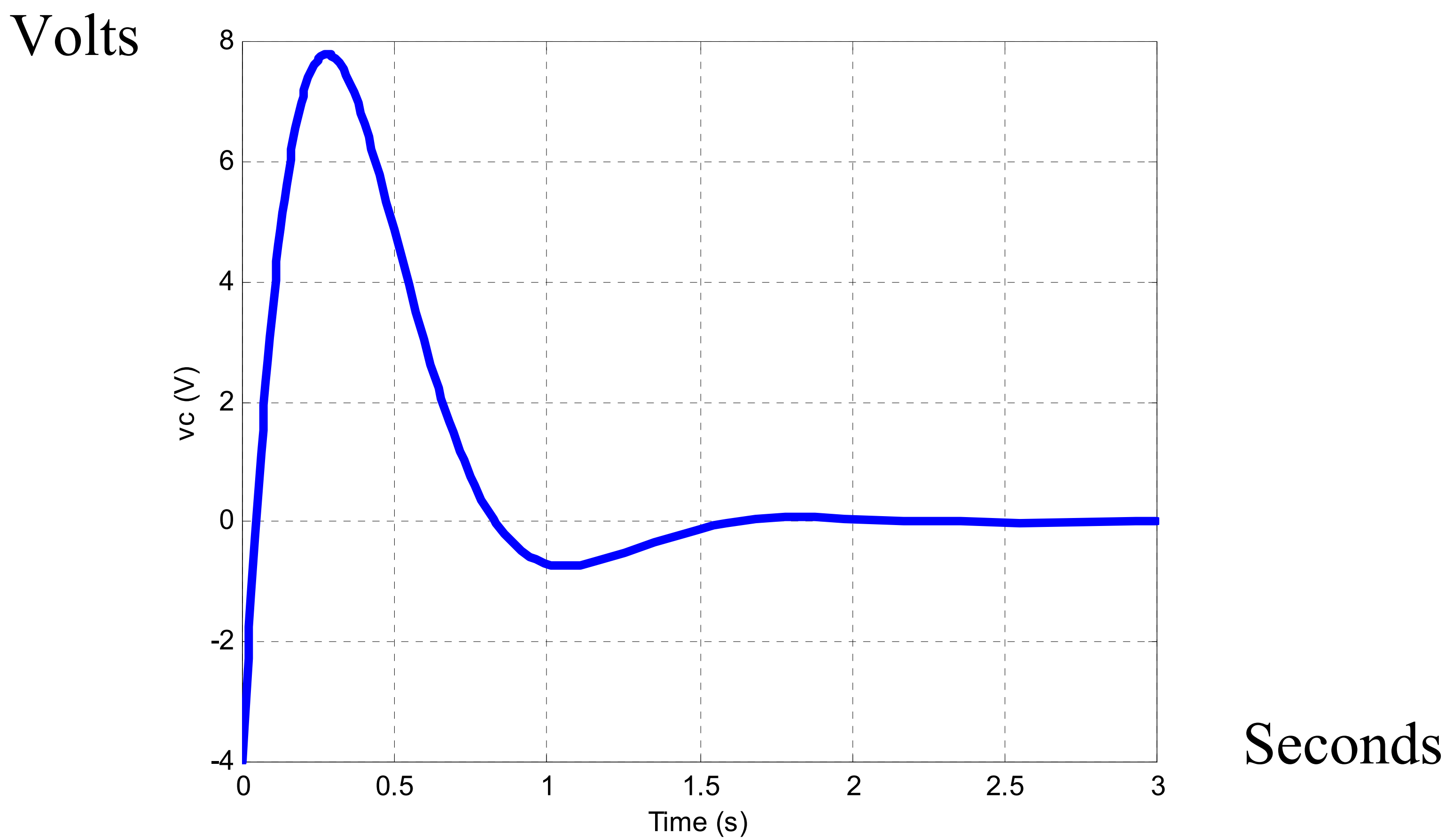


Amps



Seconds

Back to Example 2: Underdamped



$$\rightarrow \quad v_c(t) = -4e^{-3t}\cos(4t) + 22e^{-3t}\sin(4t) \ \text{ V} \qquad t > 0$$

The Forced Response

Recall: $x(t) = x_n + x_f$

Solves: $$\frac{d^2 x(t)}{dt} + a_1 \frac{dx}{dt} + a_2 x = f(t)$$

<u>Where:</u>

→ x_n is the homogeneous solution or the natural response

→ x_f is the particular solution or the forced response

How is x_f determined? → Guessing method.

Common Guesses for x_f

$f(t)$	$x_f(t)$
k	A
t	$At + B$
t^2	$At^2 + Bt + C$
e^{at}	Ae^{at}
$\sin(bt), \cos(bt)$	$A\sin(bt) + B\cos(bt)$
$e^{at}\sin(bt), e^{at}\cos(bt)$	$e^{at}(A\sin(bt) + B\cos(bt))$

Example: Forced Response

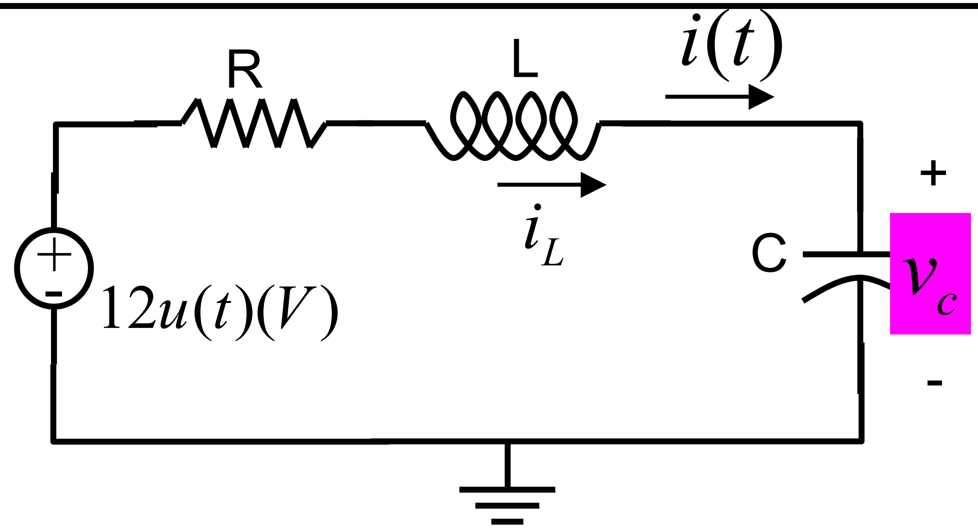
R=6Ω

L=1H

C=0.04F

$$i_L(0) = 4A$$

$$v_c(0) = -4V$$



<u>KVL</u> $$Ri + L\frac{di}{dt} + v_c = 12 \qquad t > 0$$

<u>KCL</u> $$i(t) = C\frac{dv_c}{dt} \rightarrow \frac{di}{dt} = C\frac{d^2v_c}{dt^2} \qquad t > 0$$

Combine two equations: $$RC\frac{dv_c}{dt} + LC\frac{d^2v_c}{dt^2} + v_c = 12$$

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{12}{LC}$$

Example: Forced Response

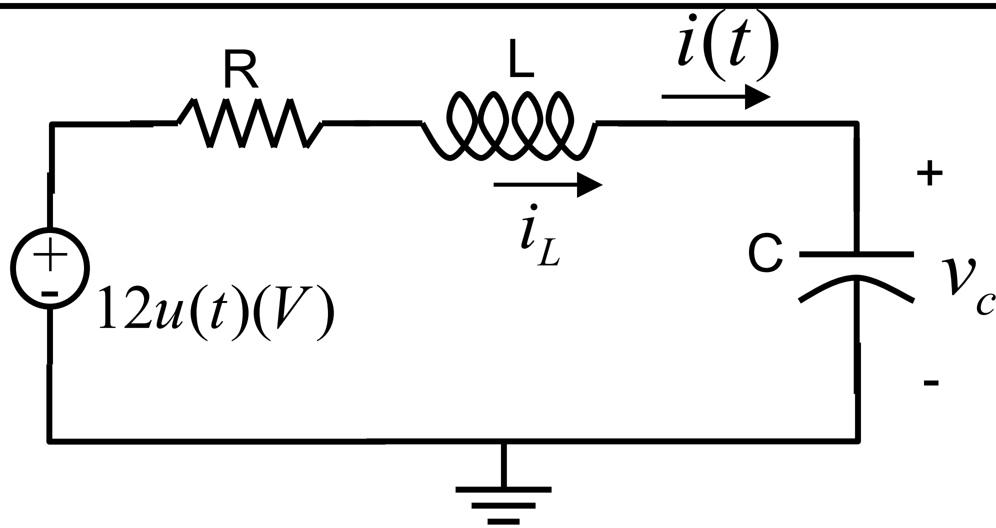
R=6Ω

L=1H

C=0.04F

$$i_L(0) = 4A$$

$$v_c(0) = -4V$$



$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{12}{LC}$$

$$\frac{d^2 v_c}{dt^2} + 6\frac{dv_c}{dt} + 25 v_c = 300$$

Now we have a forcing function.

Same as Lecture 8

$$v_c(t) = v_f + v_n$$

Example: Forced Response

The natural or homogeneous solution has the **form (see Lecture 8)**:

$$v_n(t) = B_1 e^{-3t}\cos(4t) + B_2 e^{-3t}\sin(4t)$$

The particular solution or forced response has the **form (see Table):**

$$v_f(t) = A$$ a constant

Therefore the total solution is:

$$v_c(t) = A + B_1 e^{-3t}\cos(4t) + B_2 e^{-3t}\sin(4t)$$

Find the three constants!

Example: Forced Response

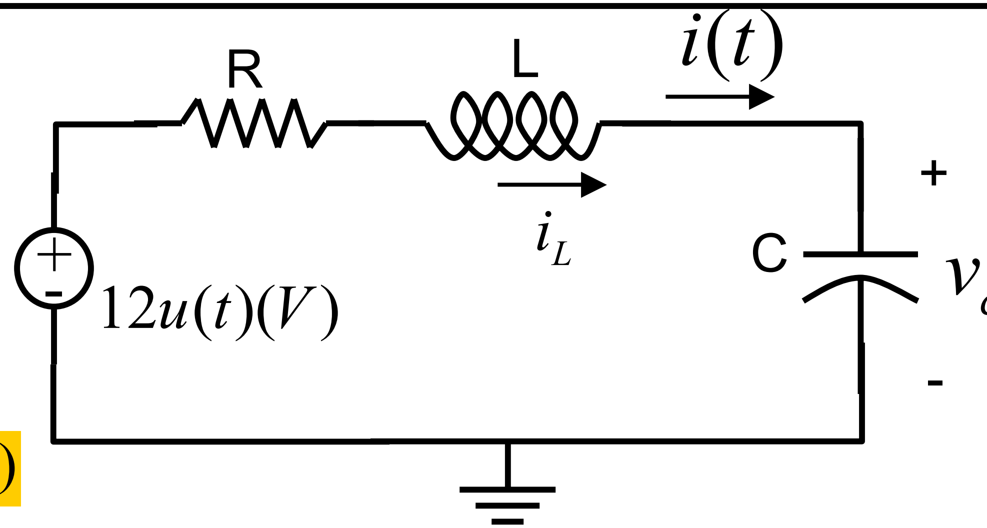
R=6Ω

L=1H

C=0.04F

$$i_L(0) = 4A$$

$$v_c(0) = -4V$$



$$v_c(t) = A + B_1 e^{-3t}\cos(4t) + B_2 e^{-3t}\sin(4t)$$

$$v_c(0) = A + B_1 = -4$$

$$v_c(\infty) = A = 12 \rightarrow B_1 = -16$$

$$\frac{dv_c(0)}{dt} = -3B_1 + 4B_2$$

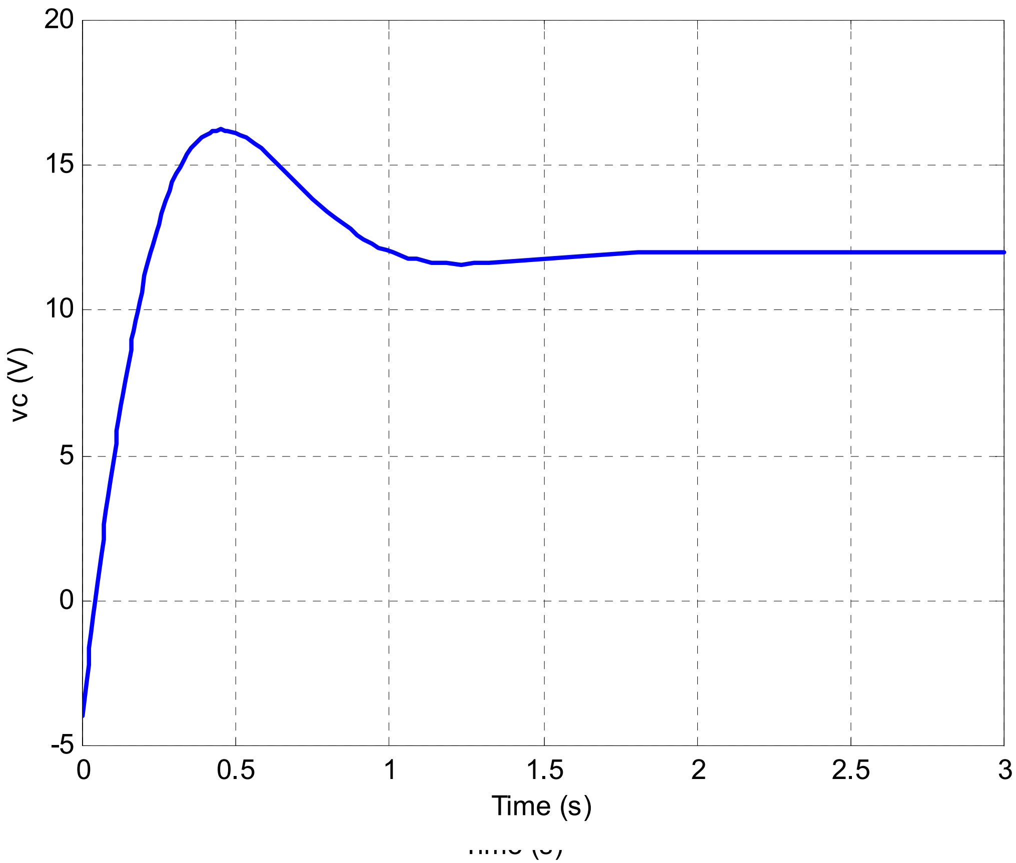
$$C\frac{dv_c(0)}{dt} = i_L(0) \rightarrow \frac{dv_c(0)}{dt} = \frac{i_L(0)}{C} = 100$$

$$\rightarrow -3B_1 + 4B_2 = 100$$

$$\rightarrow B_2 = 13$$

Example: Forced Response

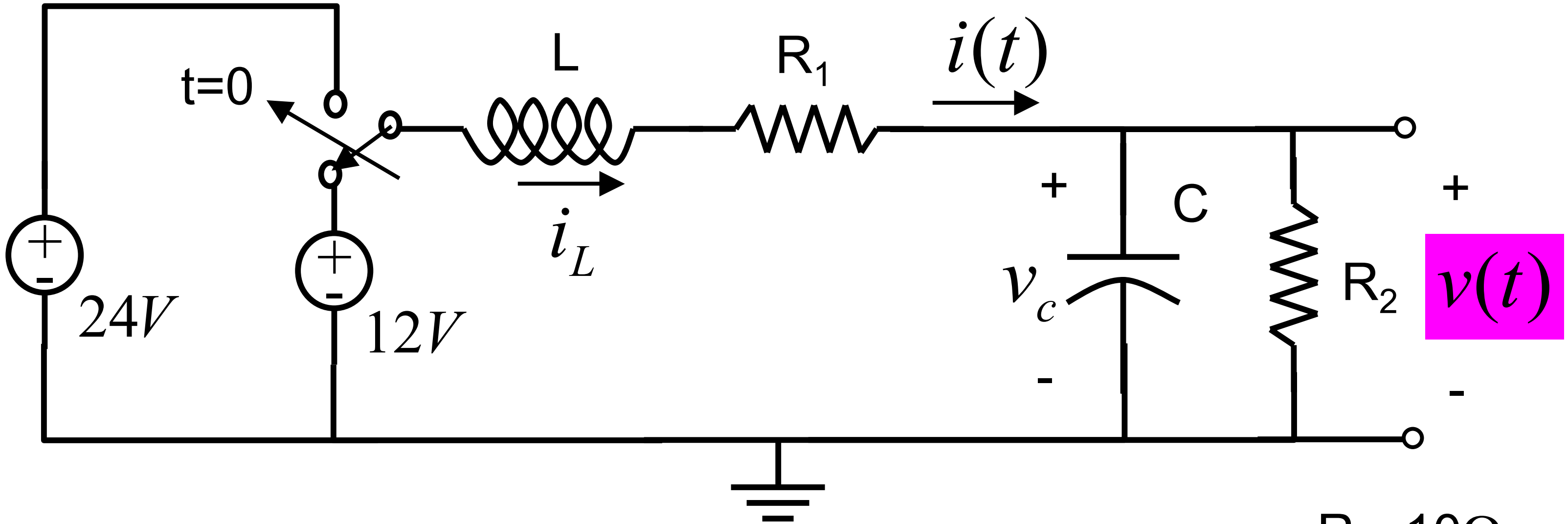
Volts



Seconds

$$v_c(t) = 12 - 16e^{-3t}\cos(4t) + 13e^{-3t}\sin(4t)$$

Example: Forced Response



$R_1=10\Omega$

$R_2=2\Omega$

L=2H

C=0.25F

t>0

KVL $$L\frac{di}{dt}+R_1 i+v=24$$

KCL $$i(t)=C\frac{dv}{dt}+\frac{v}{R_2}$$

Combining equations $$\frac{dv^2}{dt^2}+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\frac{dv}{dt}+\left(\frac{R_1+R_2}{R_2LC}\right)v=\frac{24}{LC}$$

Example: Forced Response

$$\frac{dv^2}{dt^2}+\left(\frac{1}{R_2C}+\frac{R_1}{L}\right)\frac{dv}{dt}+\left(\frac{R_1+R_2}{R_2LC}\right)v=\frac{24}{LC}$$

$$\frac{dv^2}{dt^2}+7\frac{dv}{dt}+12v=48$$

$$v(t)=v_f+v_n$$

v_n → Solution of homogeneous equation

v_f → Forced response (constant in this case)

Homogeneous equation ⇒ $$\frac{dv^2}{dt^2}+7\frac{dv}{dt}+12v=0$$

Example: Forced Response

Homogeneous equation → $$\frac{dv^2}{dt^2}+7\frac{dv}{dt}+12v=0$$

Characteristic equation → $$s^2+7s+12=0$$

Roots → $\lambda_1=-3$ $\lambda_2=-4$ → **Overdamped Response**

The natural response has the form: $$v_n=C_1e^{-3t}+C_2e^{-4t}$$

The total response has the form: $$v(t)=A+C_1e^{-3t}+C_2e^{-4t}$$

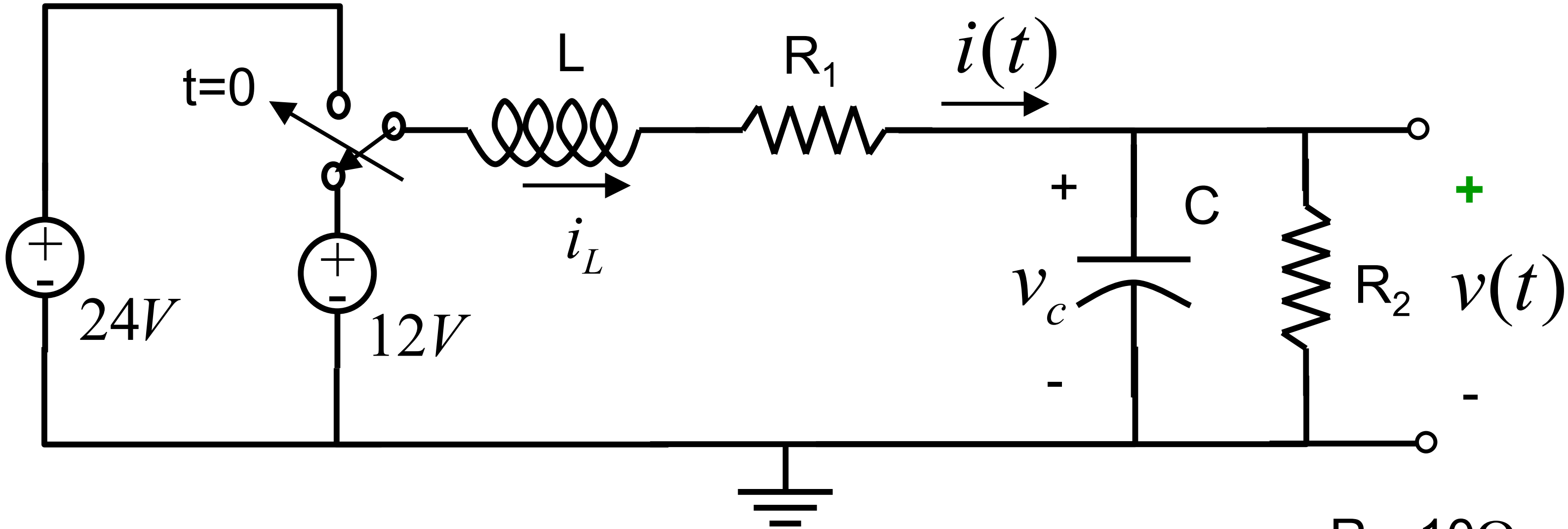
Find constants:

At steady-state derivatives are zero (or use circuit)

$$\frac{dv^2}{dt^2}+7\frac{dv}{dt}+12v=48$$

$$v(\infty)=A=4$$

Example: Forced Response



$$v(t) = A + C_1 e^{-3t} + C_2 e^{-4t}$$

$R_1=10\Omega$

$R_2=2\Omega$

L=2H

C=0.25F

☆ $v(\infty) = A = 4$ **Switch in position 2**

🕑 $v(0) = A + C_1 + C_2 = 2$ **Switch in position 1**

$$\frac{dv(0)}{dt} = -3C_1 - 4C_2$$

Top equation

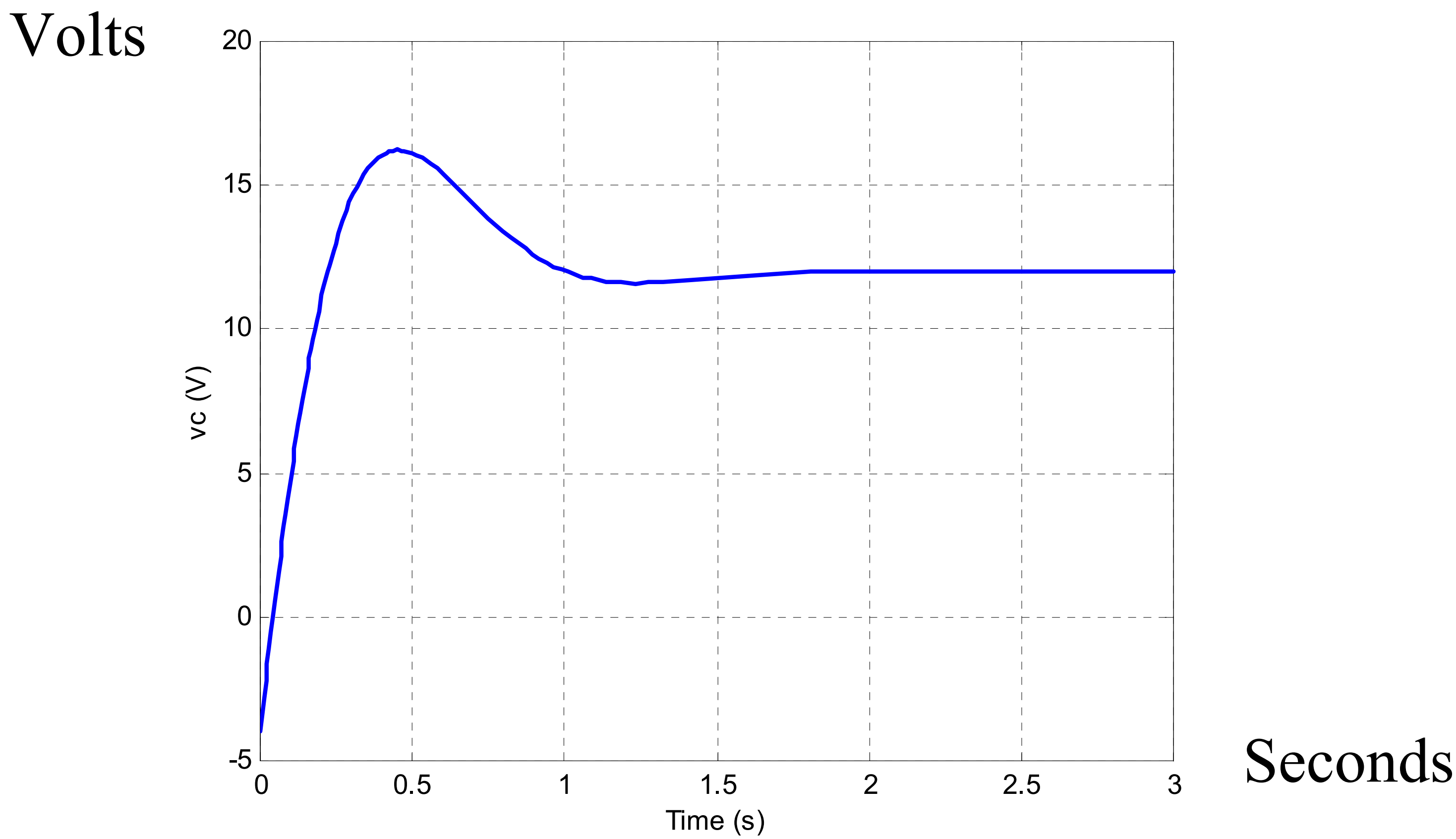
$$\frac{dv(0)}{dt} = \frac{i(0)}{C} - \frac{v(0)}{CR_2} = 0$$

KCL at + output

↑ 🕑 $3C_1 - 4C_2 = 0$

↑ $C_1 = -8 \quad C_2 = 6$

Example: Forced Response



$$v(t) = 4 - 8e^{-3t} + 6e^{-4t}$$