ECSE 210: Circuit Analysis

Lecture #8: Second Order Circuits

Conventions

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = 0$$

→ Homogeneous Equation

$$a_1 = 2\alpha \qquad a_2 = \omega_o^2$$

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_o^2x(t) = 0$$

$$\underbrace{s^2 + 2\alpha s + \omega_o^2 = 0}$$

Characteristic equation

Conventions

$$s^2 + 2\alpha s + \omega_o^2 = 0 \longrightarrow \text{Characteristic equation}$$

The roots of the characteristic equation are:

$$\lambda_1, \lambda_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

→ α is called the exponential ***damping coefficient.***
→ ω_o is called the ***undamped natural frequency***.

→ The roots of the characteristic polynomial λ_1 and λ_2 are called the natural frequencies of the circuit because they determine the natural (unforced) response.

Three Solution Types

→ Type 1: ***Overdamped***

$\alpha > \omega_o$ → λ_1 and λ_2 are ***real*** and ***unequal***

$$x_n(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

?

→ The natural response is the sum of two decaying exponentials

→ Type 2: ***Critically damped***

$\alpha = \omega_o$ → λ_1 and λ_2 are ***real*** and ***equal***

$$\lambda_1 = \lambda_2 = -\alpha$$

$$x_n(t) = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t}$$

Three Solution Types

→ Type 3: ***Underdamped***

$\alpha < \omega_o$ → λ_1 and λ_2 are ***complex*** and ***unequal***

$$\lambda_1, \lambda_2 = -\alpha \pm j\sqrt{\omega_o^2 - \alpha^2} = -\alpha \pm j\omega_d$$

$$x_n(t) = C_1 e^{-\alpha t} e^{j\omega_d t} + C_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$x_n(t) = e^{-\alpha t}\left(C_1 e^{j\omega_d t} + C_2 e^{-j\omega_d t}\right)$$

→ ω_d is called the ***damped natural frequency***

→ C_1 and C_2 can be complex.

Three Solution Types

→ Type 3: ***Underdamped***

$$x_n(t) = e^{-\alpha t}\left(C_1 e^{j\omega_d t} + C_2 e^{-j\omega_d t}\right)$$

Euler ⇒ $e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t$

$$x_n(t) = e^{-\alpha t}\left(C_1 \cos \omega_d t + jC_1 \sin \omega_d t + C_2 \cos \omega_d t - jC_2 \sin \omega_d t\right)$$

$$x_n(t) = e^{-\alpha t}\left[\left(C_1 + C_2\right)\cos \omega_d t + j\left(C_1 - C_2\right)\sin \omega_d t\right]$$

Three Solution Types

→ Type 3: ***Underdamped***

$$x_n(t) = e^{-\alpha t}\left[\left(C_1 + C_2\right)\cos\omega_d t + j\left(C_1 - C_2\right)\sin\omega_d t\right]$$

$$x_n(t) = e^{-\alpha t}\left[B_1 \cos\omega_d t + B_2 \sin\omega_d t\right]$$

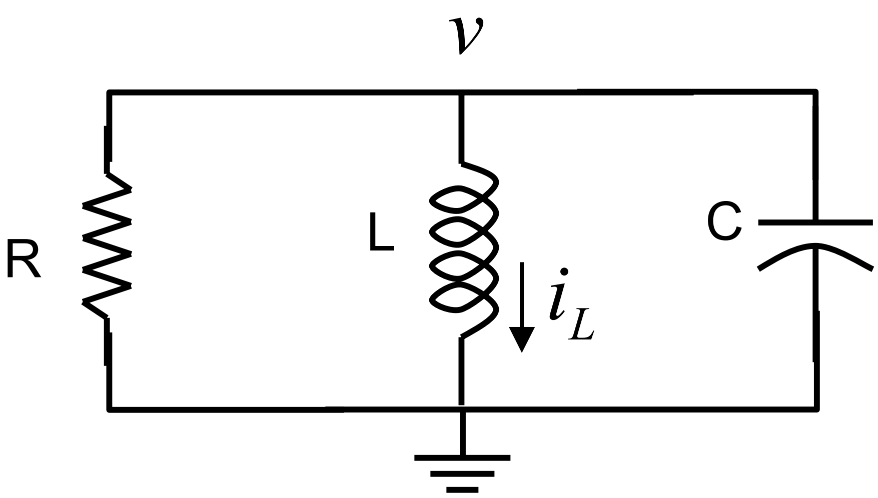
→ B_1 and B_2 are arbitrary constants.

→ For B_1 and B_2 **to be real**, C_1 and C_2 must be complex conjugates. **This is always the case for circuits.**

→ The natural underdamped response is oscillatory with an exponentially decaying magnitude.

Example: Overdamped

Find *v(t)* and *i_L(t)* .



R=2Ω

L=5H

C=0.2F

$$i_L(0) = -1A$$

$$v(0) = 4V$$

$\Uparrow$

Given two boundary conditions

KCL

$$\frac{v}{R} + \frac{1}{L}\int_0^t v(\tau)d\tau + i_L(0) + C\frac{dv}{dt} = 0$$

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Example: Overdamped

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$\frac{d^2v}{dt^2} + 2.5\frac{dv}{dt} + v = 0$$

Recall

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_o^2 x(t) = 0$$

Damping Coefficient: $\alpha = \frac{1}{2RC} = 1.25$

Resonant Frequency: $\omega_o = \frac{1}{\sqrt{LC}} = 1$

$\alpha > \omega_o$ ⇨ Overdamped

Example: Overdamped

$$\frac{d^2v}{dt^2} + 2.5\frac{dv}{dt} + v = 0$$

Characteristic equation: $s^2 + 2.5s + 1 = 0$

Roots: $\lambda_1, \lambda_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$

$$\underbrace{\lambda_1 = -2 \qquad \lambda_2 = -0.5}$$

Real and Unequal → **Overdamped Response**

$$v(t) = C_1 e^{-2t} + C_2 e^{-0.5t}$$

Example: Overdamped

$$v(t) = C_1 e^{-2t} + C_2 e^{-0.5t}$$

Find coefficients C_1 and C_2

$$v(0) = v_c(0) = C_1 + C_2 = 4$$

First boundary condition

→ Need one more equation!

→ Use $i_L(0)$

<u>KCL</u>

$$C\frac{dv}{dt} + \frac{v}{R} + i_L = 0$$

From the circuit

Same as 1st equation

→ $$\frac{dv(0)}{dt} = -\frac{1}{RC}v(0) - \frac{1}{C}i_L(0) = -5$$

Second boundary condition

Example: Overdamped

$$v(t) = C_1 e^{-2t} + C_2 e^{-0.5t}$$

Next use the proposed solution

$$\frac{dv(t)}{dt} = -2C_1 e^{-2t} - 0.5C_2 e^{-0.5t}$$

$$\frac{dv(0)}{dt} = -2C_1 - 0.5C_2 = -5$$

Remember $C_1 + C_2 = 4$

→ $C_1 = 2$ $C_2 = 2$

$$v(t) = 2e^{-2t} + 2e^{-0.5t} \qquad ; t > 0$$

Example: Overdamped

Find the inductor current for t>0.

Linear circuit → i_L(t) has the same form:

$$i_L(t) = C_1 e^{-2t} + C_2 e^{-0.5t}$$

$$i_L(0) = C_1 + C_2 = -1$$

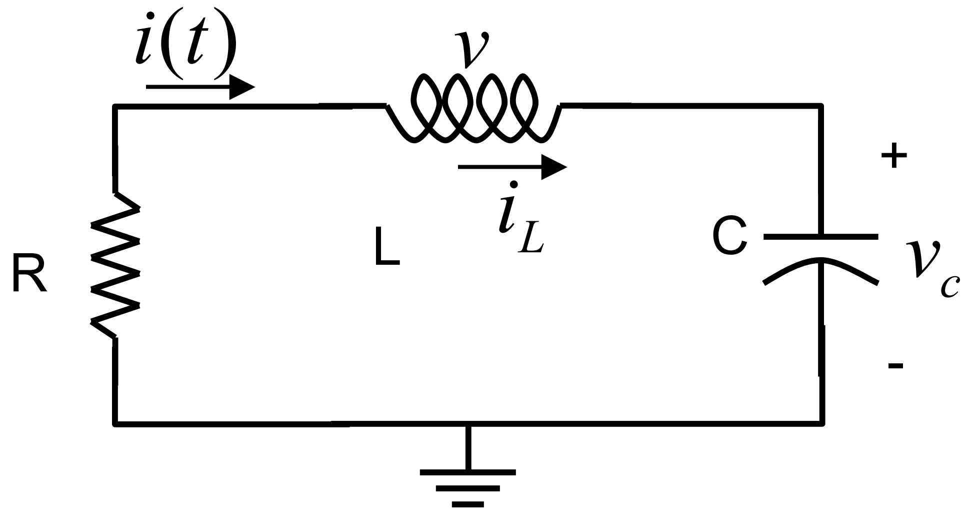
$$v(0) = L\frac{di_L(0)}{dt} = L(-2C_1 - 0.5C_2) = 4$$

Two boundary conditions

→ $C_1 = -0.2 \quad C_2 = -0.8$

$$i_L(t) = -0.2e^{-2t} - 0.8e^{-0.5t} \qquad ; t > 0$$

Example 2: Underdamped



R=6Ω

L=1H

C=0.04F

$$i_L(0) = 4A$$

$$v_c(0) = -4V$$

Find *i(t)* and *v_c(t)*

<u>KVL</u>

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0) = 0$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

Example 2: Underdamped

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

Recall

$$\frac{d^2 x(t)}{dt^2} + 2\alpha \frac{dx(t)}{dt} + \omega_o^2 x(t) = 0$$

$$\frac{d^2 i}{dt^2} + 6\frac{di}{dt} + 25i = 0$$

Damping Coefficient: $\alpha = \frac{R}{2L} = 3$

Resonant Frequency: $\omega_o = \frac{1}{\sqrt{LC}} = 5$

$\alpha < \omega_o$ ➡ underdamped

Example 2: Underdamped

$$\frac{d^2 i}{dt^2} + 6\frac{di}{dt} + 25i = 0$$

Characteristic equation: $s^2 + 6s + 25 = 0$

Roots: $\lambda_1, \lambda_2 = -\alpha \pm j\sqrt{\omega_o^2 - \alpha^2}$

$$\underbrace{\lambda_1 = -3 + j4 \qquad \lambda_2 = -3 - j4}$$

Complex → Underdamped Response

$$i(t) = B_1 e^{-3t}\cos(4t) + B_2 e^{-3t}\sin(4t)$$

Example 2: Underdamped

☆ $i(t) = B_1 e^{-3t}\cos(4t) + B_2 e^{-3t}\sin(4t)$

$$i(0) = B_1 = i_L(0) = 4$$

$$\frac{di(t)}{dt} = -3B_1 e^{-3t}\cos(4t) - 4B_1 e^{-3t}\sin(4t)$$

$$-3B_2 e^{-3t}\sin(4t) + 4B_2 e^{-3t}\cos(4t)$$

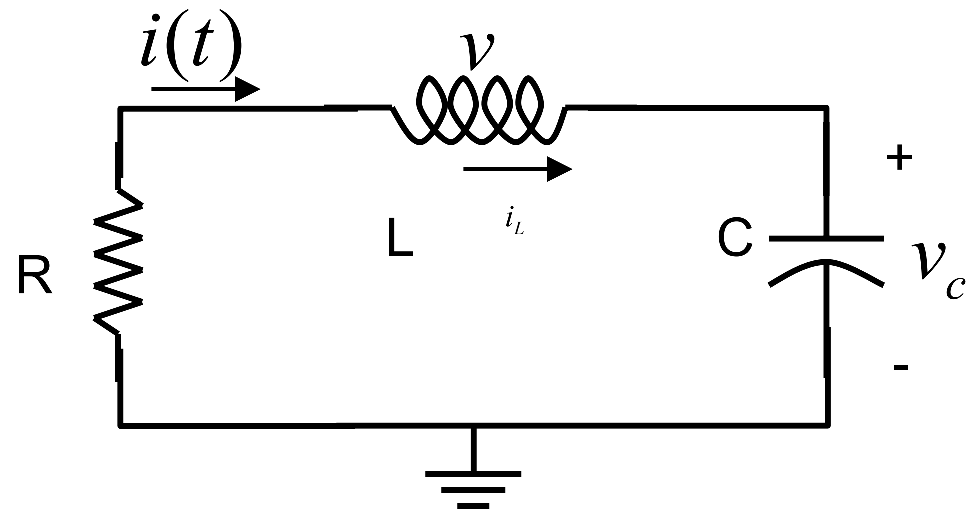
$$\frac{di(0)}{dt} = -3B_1 + 4B_2$$

Example 2: Underdamped

<u>KVL</u>

$$Ri + L\frac{di}{dt} + v_c = 0$$

$$\frac{di(0)}{dt} = -\frac{R}{L}i(0) - \frac{v_c(0)}{L}$$



$$\frac{di(0)}{dt} = -24 + 4 = -20$$ **from circuit**

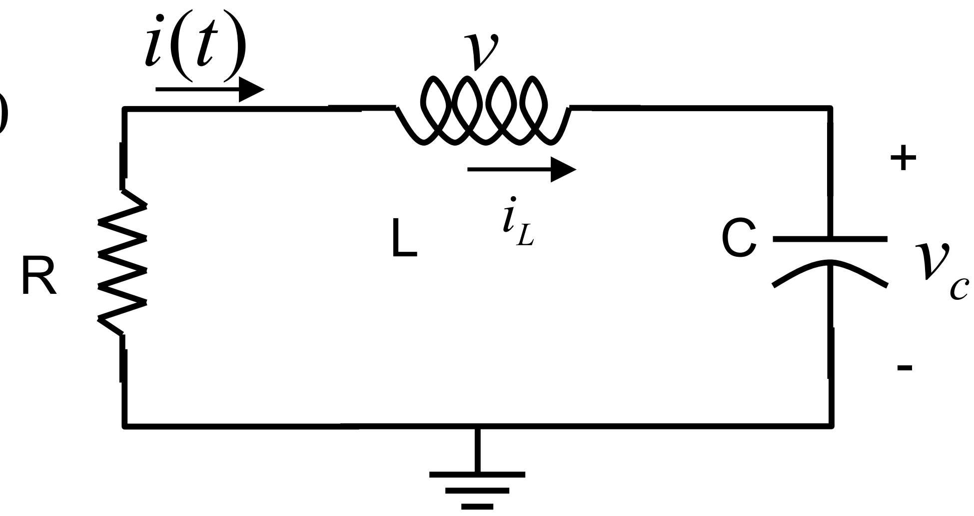
$$\frac{di(0)}{dt} = -3B_1 + 4B_2 = -20 \rightarrow \begin{cases} B_2 = -2 \\ B_1 = 4 \end{cases}$$

from differential equation

$$i(t) = 4e^{-3t}\cos(4t) - 2e^{-3t}\sin(4t)$$

Example 2: Underdamped

Find the capacitor voltage for t>0



$$v_c(t) = B_1 e^{-3t}\cos(4t) + B_2 e^{-3t}\sin(4t)$$

$$v_c(0) = B_1 = -4$$

$$\frac{dv_c(0)}{dt} = -3B_1 + 4B_2 = \frac{i(0)}{C} = 100 \quad \Rightarrow \quad B_2 = 22$$

$$v_c(t) = -4e^{-3t}\cos(4t) + 22e^{-3t}\sin(4t)$$