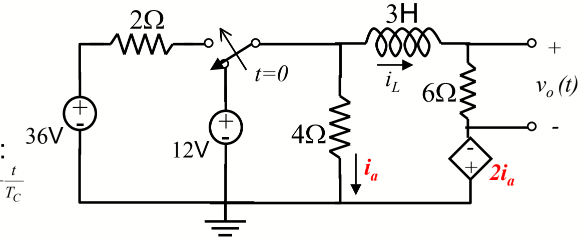
ECSE 210: Circuit Analysis

Lecture #7: First and Second Order Circuits

RL/RC Circuits With a Controlled Source

Find $v_o(t)$ for $t>0$



Step 1:

Assume solution:

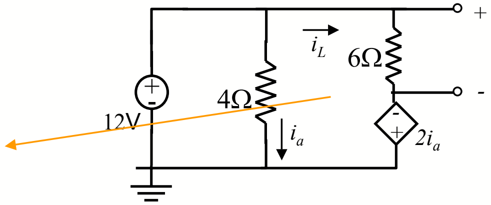
$$v_o(t) = C_1 + C_2 e^{-\frac{t}{T_C}}$$

Step 2:

Find $i_L(0^-)$

$$i_a = \frac{12V}{4\Omega} = 3A$$

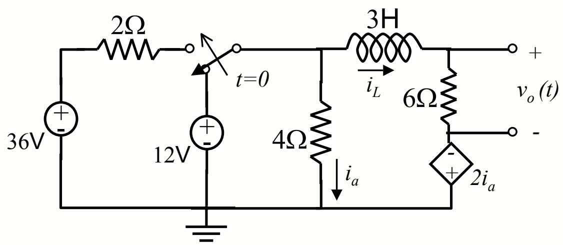
$$i_L(0^-) = \frac{12V + 2i_a}{6\Omega} = 3A$$



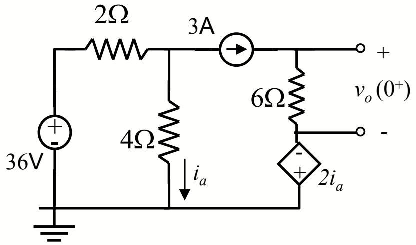
RL/RC Circuits with a Controlled Source

Step 3:

Find $v_o(0^+)$



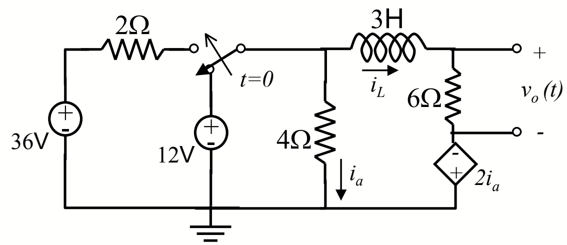
$$v_o(0^+) = (3A)(6\Omega) = 18V$$



RL/RC Circuits with a Controlled Source

Step 4:

Find $v_o(\infty)$



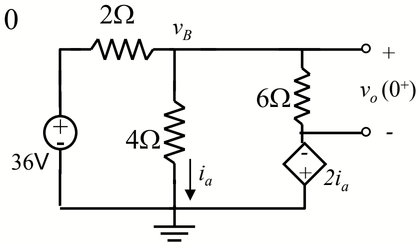
KCL at node B:

$$\frac{v_B - 36V}{2\Omega} + \frac{v_B}{4\Omega} + \frac{v_B + 2v_B/4\Omega}{6\Omega} = 0$$

$$v_B = 18V$$

$$i_a = \frac{18V}{4\Omega} = 4.5A$$

$$v_o(\infty) = 2i_a + v_B = 27V$$



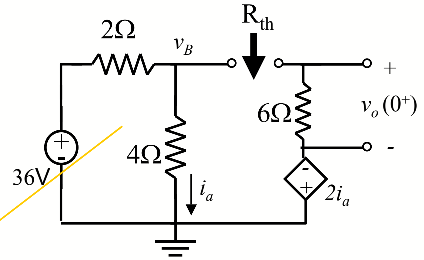
RL/RC Circuits with a Controlled Source

Step 5:

Find C_1 and C_2

$$C_1 = v_o(\infty) = 27$$

$$C_2 = v_o(0^+) - v_o(\infty) = -9$$



Step 6:

Find T_c

$$i_{sc} = \frac{v_o(\infty)}{6\Omega} = 4.5A$$

$$v_{oc} = 2i_a + 4i_a$$

$$i_a = \frac{36V}{2\Omega + 4\Omega} = 6A$$

$$v_{oc} = 36V$$

$$R_{th} = \frac{v_{oc}}{i_{sc}} = 8\Omega$$

$$T_C = \frac{L}{R_{th}} = 3/8s$$

$$v(t) = 27 - 9e^{-\frac{8}{3}t} \ (V)$$

Sinusoidal Input

- Response of a first order circuit:

$$x = x_f + x_n$$

x_n: Natural response

x_f: Forced response

- For a periodic signal:

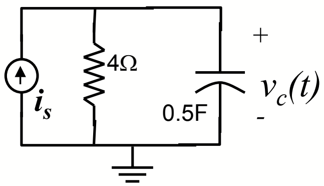
$$f(t+T) = f(t)$$

An example periodic input: sinusoid

Sinusoidal Input: An ODE Approach

$$i_s(t) = 10\sin(2t)u(t)$$

$$v_c(t) = v_f + v_n$$



$$0.5\frac{dv_c}{dt} + \frac{v_c}{4} = 10\sin(2t)$$

The form of the natural response does not depend on the input.
What is the forced response due do a sinusoidal input?

$$v_f = A\sin(2t) + B\cos(2t)$$

$$\frac{dv_f}{dt} = 2A\cos(2t) - 2B\sin(2t)$$

Sinusoidal Input

Substitute into ODE:

$$(A\cos(2t) - B\sin(2t)) + \frac{1}{4}(A\sin(2t) + B\sin(2t)) = 10\sin(2t)$$

Equate coefficients of sine and cosine:

$$\frac{A}{4} - B = 10 \qquad A + \frac{B}{4} = 0$$

$$v_f = A\sin(2t) + B\cos(2t)$$

$$v_f = \frac{40}{17}\sin(2t) - \frac{160}{17}\cos(2t)$$

Sinusoidal Input

Original equation: $0.5\frac{dv_c}{dt}+\frac{v_c}{4}=10\sin(2t)$

The natural response is the solution of the homogeneous equation: $$0.5\frac{dv_c}{dt}+\frac{v_c}{4}=0$$

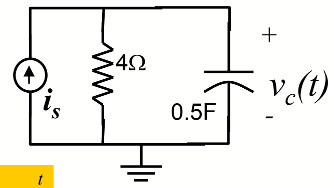
Assume: $v_n = De^{-st}$

Plug into homogeneous equation:

$$-0.5sDe^{-st}+\frac{1}{4}De^{-st}=0$$

$$\left(-0.5s+\frac{1}{4}\right)De^{-st}=0 \qquad s=\frac{1}{2}$$

Sinusoidal Input



$$v_c(t)=\frac{40}{17}\sin(2t)-\frac{160}{17}\cos(2t)+De^{-\frac{t}{2}}$$

$$v_c(0^+)=-\frac{160}{17}+D=0 \qquad D=\frac{160}{17}$$

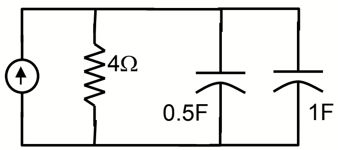
Apply boundary condition

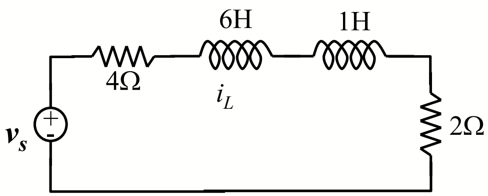
$$v_c(t)=\frac{40}{17}\sin(2t)-\frac{160}{17}\cos(2t)+\frac{160}{17}e^{-\frac{t}{2}} \quad \text{Volts}$$

Second Order Circuits

- → Contains two storage elements
- → Performance described by a second order ODE
- → Boundary conditions on both C and L must hold!

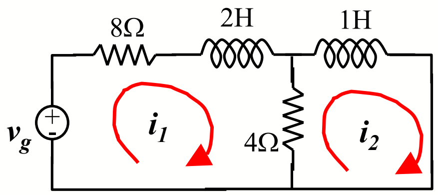
Are these circuits second order?





Second Order Circuits

Mesh equations:



☆ $2\frac{di_1}{dt}+(8+4)i_1-4i_2=v_g$

✲ $-4i_1+\frac{di_2}{dt}+4i_2=0 \quad \Rightarrow \quad i_1=\frac{1}{4}\left(\frac{di_2}{dt}+4i_2\right)$

Taking the derivative, $\frac{di_1}{dt}=\frac{1}{4}\left(\frac{d^2i_2}{dt^2}+4\frac{di_2}{dt}\right)$

Second Order Circuits

☆ $$2\frac{di_1}{dt}+(8+4)i_1-4i_2=v_g$$

⌘ $$\frac{di_1}{dt}=\frac{1}{4}\left(\frac{d^2i_2}{dt^2}+4\frac{di_2}{dt}\right)$$

⌘ $$i_1=\frac{1}{4}\left(\frac{di_2}{dt}+4i_2\right)$$

Substitute ⌘ into ☆

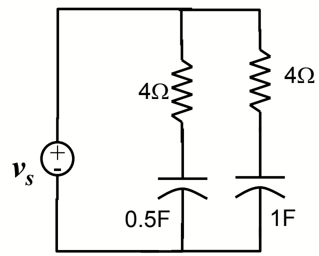
$$\frac{1}{2}\left(\frac{d^2i_2}{dt^2}+4\frac{di_2}{dt}\right)+3\left(\frac{di_2}{dt}+4i_2\right)-4i_2=v_g$$

$$\frac{d^2i_2}{dt^2}+10\frac{di_2}{dt}+16i_2=2v_g$$ ➡ Second order ODE

Second Order Circuits

- Is this a second order circuit?
- Note that there are two uncoupled branches.

See J, J, H &S Section 7.1



Second Order Circuits

$$\frac{d^2i_2}{dt^2}+10\frac{di_2}{dt}+16i_2=2v_g$$ ➡ Second order ODE

A generalized linear second order ODE:

$$\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_2x(t)=f(t)$$

$$x=x_f+x_n$$

x_n: Natural response

x_f: Forced response

Second Order Circuits

The natural response is found from the general solution of the homogeneous equations:

$$\frac{d^2x(t)}{dt^2}+a_1\frac{dx(t)}{dt}+a_2x(t)=0$$

Suppose $x_n=De^{st}$

$$s^2De^{st}+a_1sDe^{st}+a_2De^{st}=0$$

$$\left(s^2+a_1s+a_2\right)De^{st}=0$$

$\left(s^2+a_1s+a_2\right)$: Characteristic polynomial

Second Order Circuits

Characteristic polynomial has two roots: λ_1 and λ_1

$$s^2 + a_1 s + a_2 = 0 \quad \Rightarrow \quad s = \lambda_1 \quad \text{or} \quad s = \lambda_2$$

$$\lambda_1, \lambda_1 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$x_n = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad ; \lambda_1 \neq \lambda_2$$

or

$$x_n = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t} \quad ; \lambda_1 = \lambda_2$$