

ECSE 210: Circuit Analysis

Lecture #7: First and Second Order Circuits

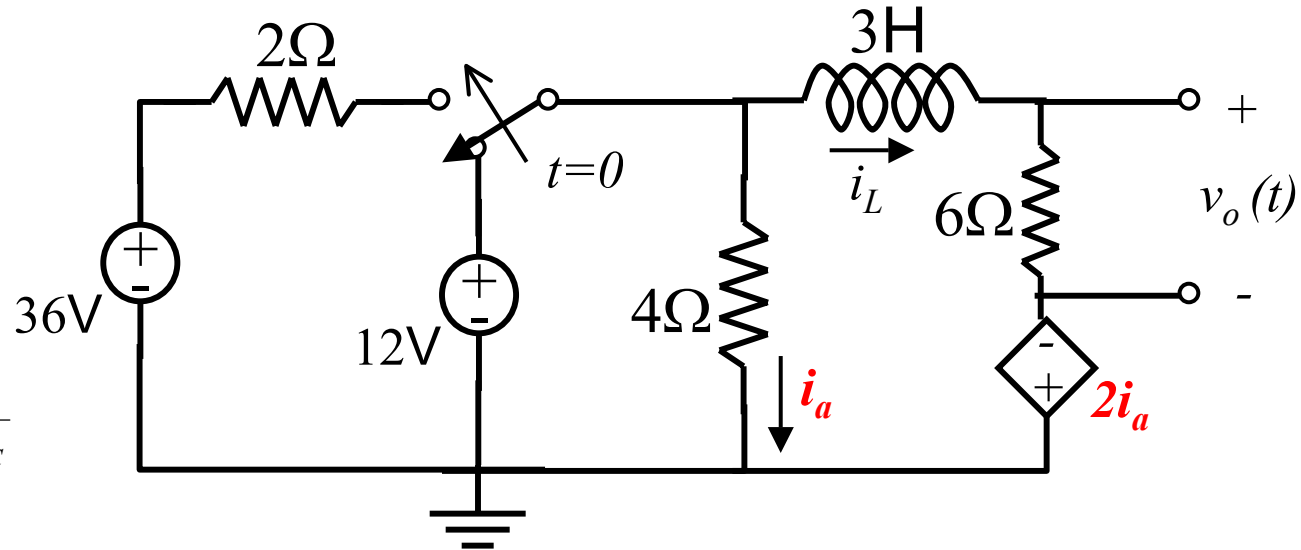
RL/RC Circuits With a Controlled Source

Find $v_o(t)$ for $t > 0$

Step 1:

Assume solution:

$$v_o(t) = C_1 + C_2 e^{-\frac{t}{T_c}}$$

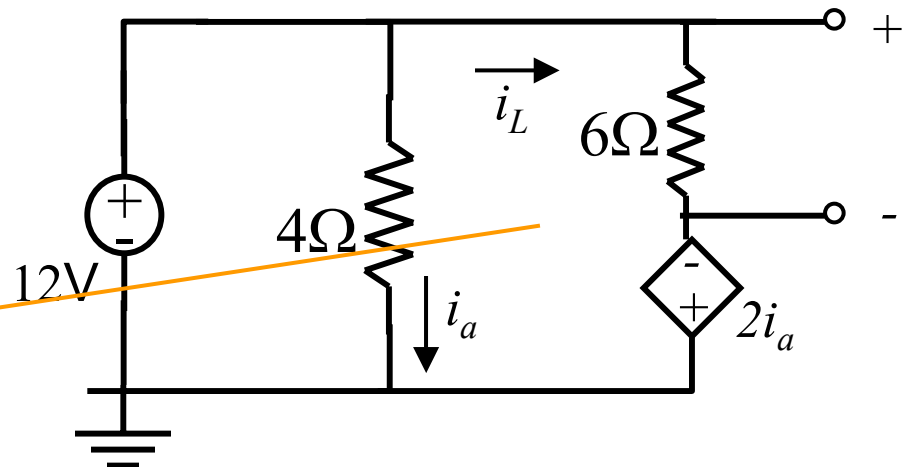


Step 2:

Find $i_L(0^-)$

$$i_a = \frac{12V}{4\Omega} = 3A$$

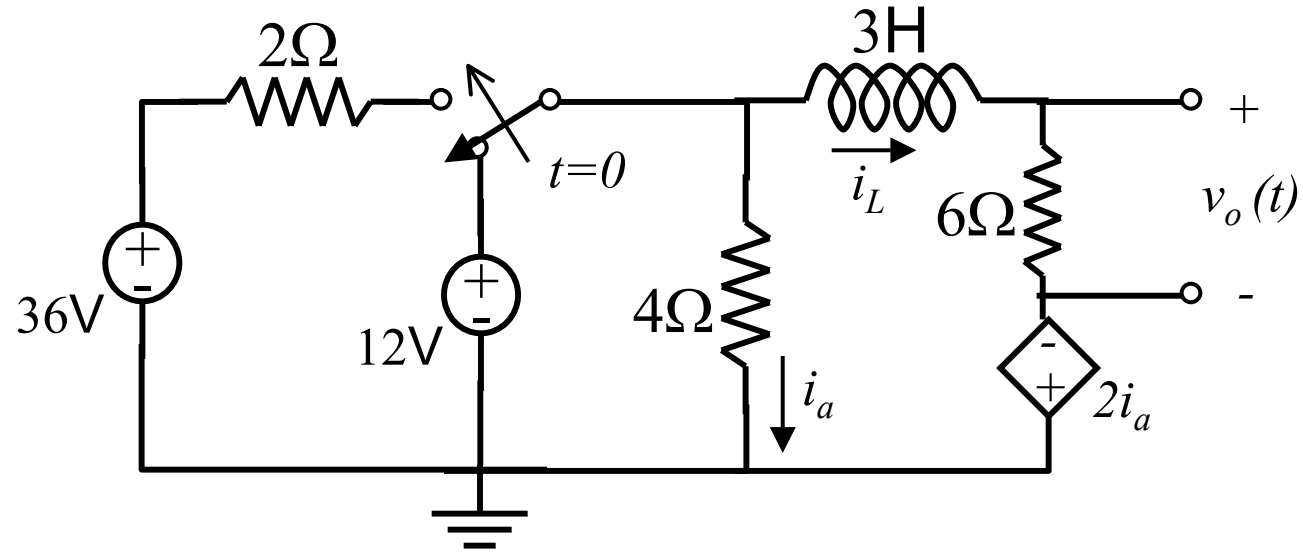
$$i_L(0^-) = \frac{12V + 2i_a}{6\Omega} = 3A$$



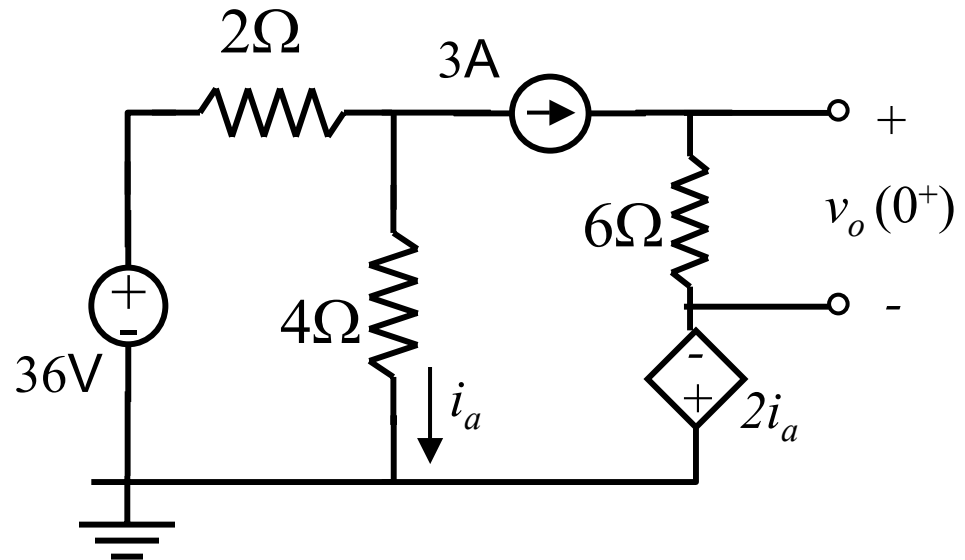
RL/RC Circuits with a Controlled Source

Step 3:

Find $v_o(0^+)$



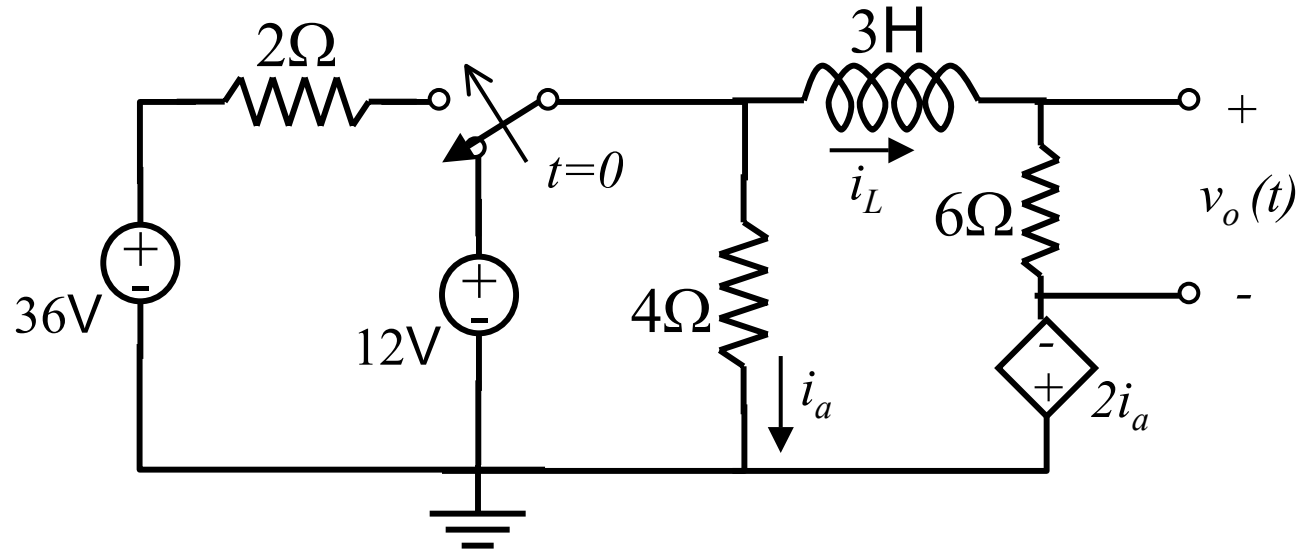
$$v_o(0^+) = (3A)(6\Omega) = 18V$$



RL/RC Circuits with a Controlled Source

Step 4:

Find $v_o(\infty)$



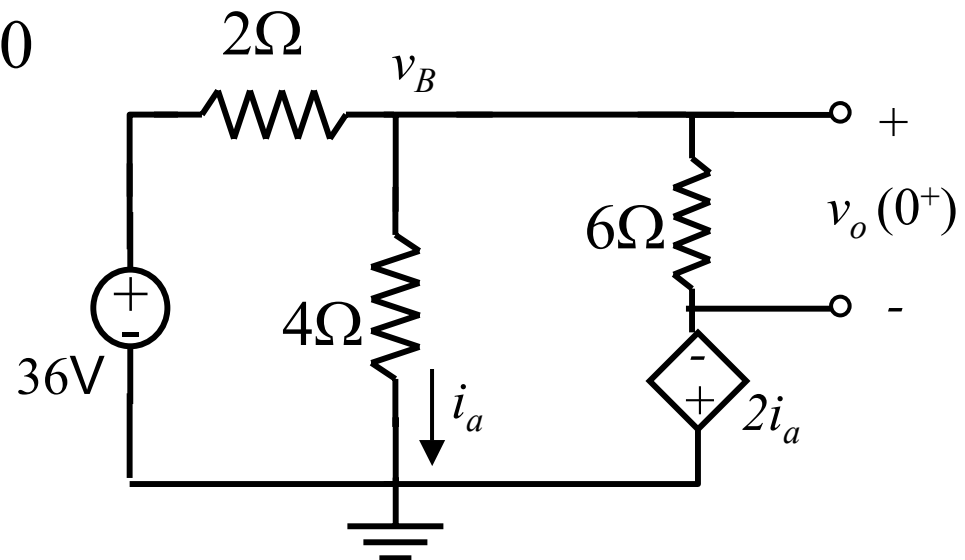
KCL at node B:

$$\frac{v_B - 36V}{2\Omega} + \frac{v_B}{4\Omega} + \frac{v_B + 2v_B/4\Omega}{6\Omega} = 0$$

$$v_B = 18V$$

$$i_a = \frac{18V}{4\Omega} = 4.5A$$

$$v_o(\infty) = 2i_a + v_B = 27V$$



RL/RC Circuits with a Controlled Source

Step 5:

Find C_1 and C_2

$$C_1 = v_o(\infty) = 27$$

$$C_2 = v_o(0^+) - v_o(\infty) = -9$$

Step 6:

Find T_c

$$i_{sc} = \frac{v_o(\infty)}{6\Omega} = 4.5A$$

$$v_{oc} = 2i_a + 4i_a$$

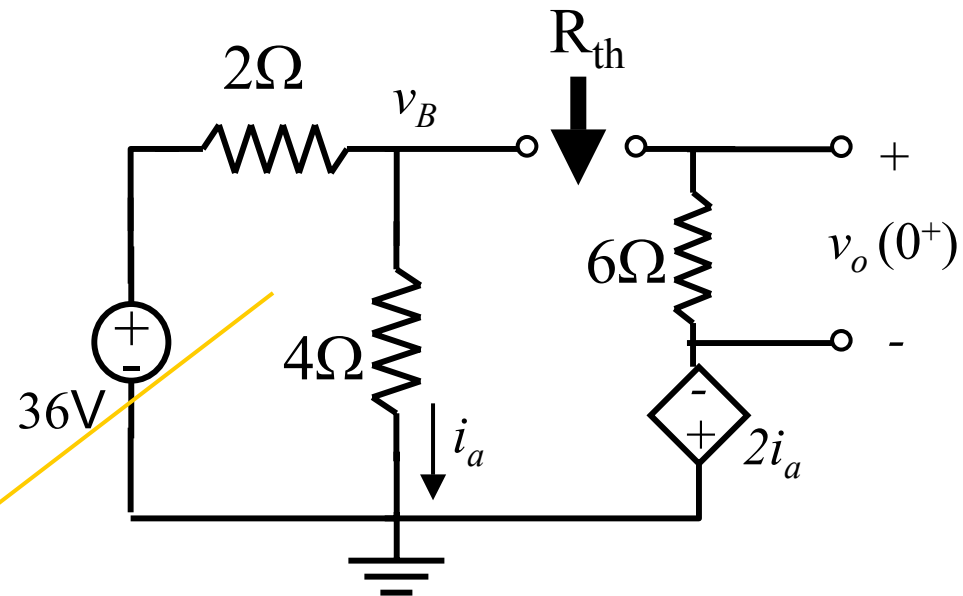
$$i_a = \frac{36V}{2\Omega + 4\Omega} = 6A$$

$$v_{oc} = 36V$$

$$R_{th} = \frac{v_{oc}}{i_{sc}} = 8\Omega$$

$$T_c = \frac{L}{R_{th}} = 3/8s$$

$$v(t) = 27 - 9e^{-\frac{8}{3}t} (V)$$



Sinusoidal Input

- Response of a first order circuit:

$$x = x_f + x_n$$

↙ Forced response ↘ Natural response

- For a periodic signal:

$$f(t + T) = f(t)$$

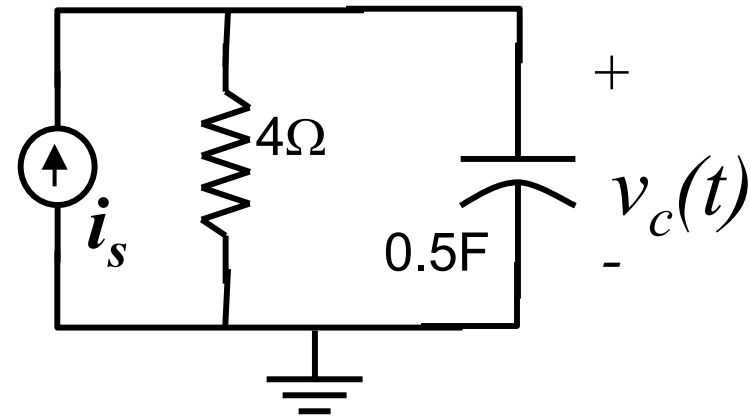
An example periodic input: sinusoid

Sinusoidal Input: An ODE Approach

$$i_s(t) = 10 \sin(2t)u(t)$$

$$v_c(t) = v_f + v_n$$

$$0.5 \frac{dv_c}{dt} + \frac{v_c}{4} = 10 \sin(2t)$$



The form of the natural response does not depend on the input.
What is the forced response due to a sinusoidal input?

$$v_f = A \sin(2t) + B \cos(2t)$$

$$\frac{dv_f}{dt} = 2A \cos(2t) - 2B \sin(2t)$$

Sinusoidal Input

Substitute into ODE:

$$(A \cos(2t) - B \sin(2t)) + \frac{1}{4}(A \sin(2t) + B \sin(2t)) = 10 \sin(2t)$$

Equate coefficients of sine and cosine:

$$\frac{A}{4} - B = 10 \qquad A + \frac{B}{4} = 0$$

$$v_f = A \sin(2t) + B \cos(2t)$$

$$v_f = \frac{40}{17} \sin(2t) - \frac{160}{17} \cos(2t)$$

Sinusoidal Input

Original equation: $0.5 \frac{dv_c}{dt} + \frac{v_c}{4} = 10 \sin(2t)$

The natural response is the solution of the homogeneous equation:

$$0.5 \frac{dv_c}{dt} + \frac{v_c}{4} = 0$$

Assume: $v_n = De^{-st}$

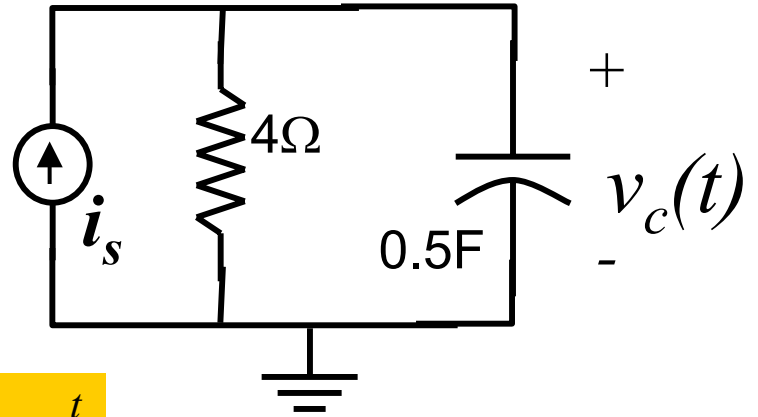
Plug into homogeneous equation:

$$-0.5sDe^{-st} + \frac{1}{4}De^{-st} = 0$$

$$\left(-0.5s + \frac{1}{4}\right)De^{-st} = 0$$

$$s = \frac{1}{2}$$

Sinusoidal Input



$$v_c(t) = \frac{40}{17} \sin(2t) - \frac{160}{17} \cos(2t) + D e^{-\frac{t}{2}}$$

$$v_c(0^+) = -\frac{160}{17} + D = 0 \quad D = \frac{160}{17}$$

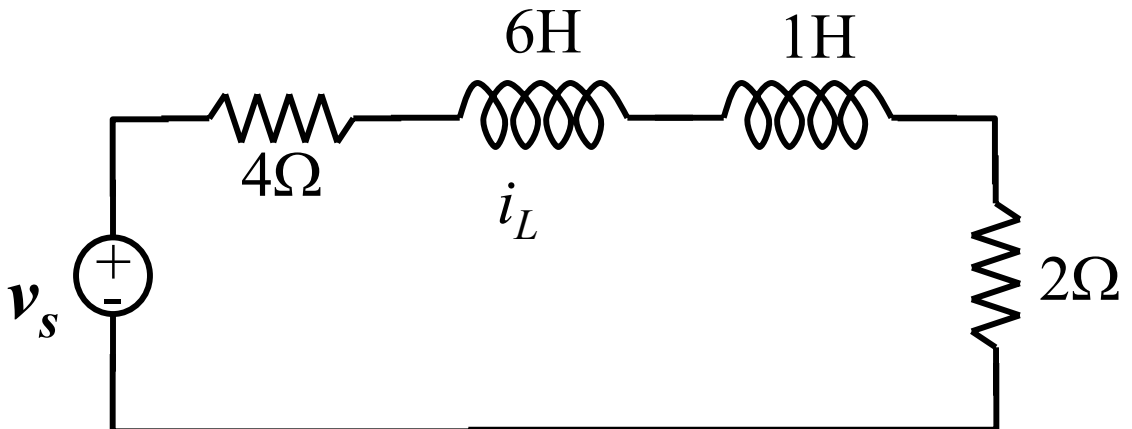
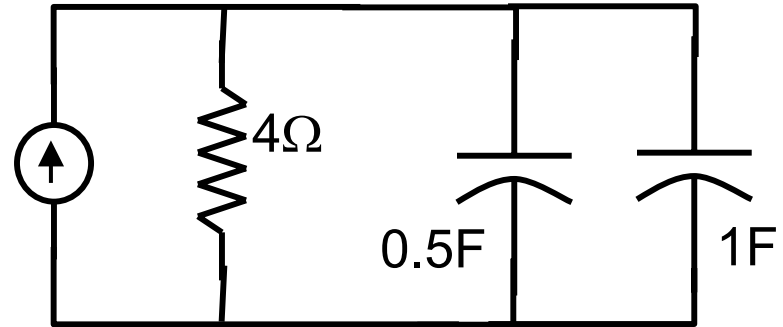
**Apply
boundary
condition**

$$v_c(t) = \frac{40}{17} \sin(2t) - \frac{160}{17} \cos(2t) + \frac{160}{17} e^{-\frac{t}{2}} \quad \text{Volts}$$

Second Order Circuits

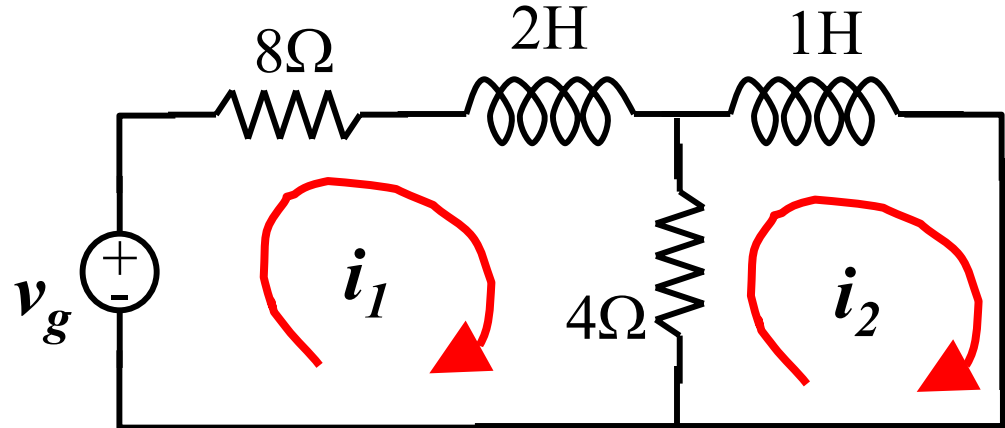
- Contains two storage elements
- Performance described by a second order ODE
- Boundary conditions on both C and L must hold!

Are these circuits second order?



Second Order Circuits

Mesh equations:



$$\star \quad 2 \frac{di_1}{dt} + (8 + 4)i_1 - 4i_2 = v_g$$

$$\star \quad -4i_1 + \frac{di_2}{dt} + 4i_2 = 0 \quad \rightarrow \quad i_1 = \frac{1}{4} \left(\frac{di_2}{dt} + 4i_2 \right)$$

$$\text{Taking the derivative,} \quad \frac{di_1}{dt} = \frac{1}{4} \left(\frac{d^2 i_2}{dt^2} + 4 \frac{di_2}{dt} \right)$$

Second Order Circuits

$$\star \quad 2 \frac{di_1}{dt} + (8 + 4)i_1 - 4i_2 = v_g$$

$$\star \quad \frac{di_1}{dt} = \frac{1}{4} \left(\frac{d^2 i_2}{dt^2} + 4 \frac{di_2}{dt} \right)$$

$$\star \quad i_1 = \frac{1}{4} \left(\frac{di_2}{dt} + 4i_2 \right)$$

Substitute
✧ into \star

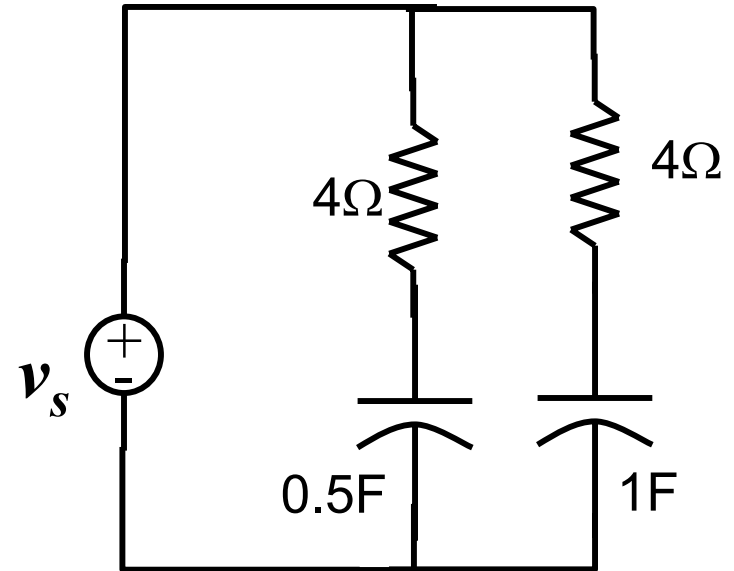
$$\frac{1}{2} \left(\frac{d^2 i_2}{dt^2} + 4 \frac{di_2}{dt} \right) + 3 \left(\frac{di_2}{dt} + 4i_2 \right) - 4i_2 = v_g$$

$$\frac{d^2 i_2}{dt^2} + 10 \frac{di_2}{dt} + 16i_2 = 2v_g \quad \rightarrow \quad \text{Second order ODE}$$

Second Order Circuits

- Is this a second order circuit?
- Note that there are two uncoupled branches.

See J, J, H & S Section 7.1



Second Order Circuits

$$\frac{d^2 i_2}{dt^2} + 10 \frac{di_2}{dt} + 16i_2 = 2v_g \quad \longrightarrow \quad \text{Second order ODE}$$

A generalized linear second order ODE:

$$\frac{d^2 x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_2 x(t) = f(t)$$

Natural response

$$x = x_f + x_n$$

Forced response

Second Order Circuits

The natural response is found from the general solution of the homogeneous equations:

$$\frac{d^2 x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_2 x(t) = 0$$

Suppose

$$x_n = De^{st}$$

$$s^2 De^{st} + a_1 s De^{st} + a_2 De^{st} = 0$$

$$\underbrace{(s^2 + a_1 s + a_2)}_{\text{Characteristic polynomial}} De^{st} = 0$$

Characteristic polynomial

Second Order Circuits

Characteristic polynomial has two roots: λ_1 and λ_2

$$s^2 + a_1s + a_2 = 0 \quad \longrightarrow \quad s = \lambda_1 \quad \text{or} \quad s = \lambda_2$$

$$\lambda_1, \lambda_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$x_n = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad ; \lambda_1 \neq \lambda_2$$

or

$$x_n = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t} \quad ; \lambda_1 = \lambda_2$$