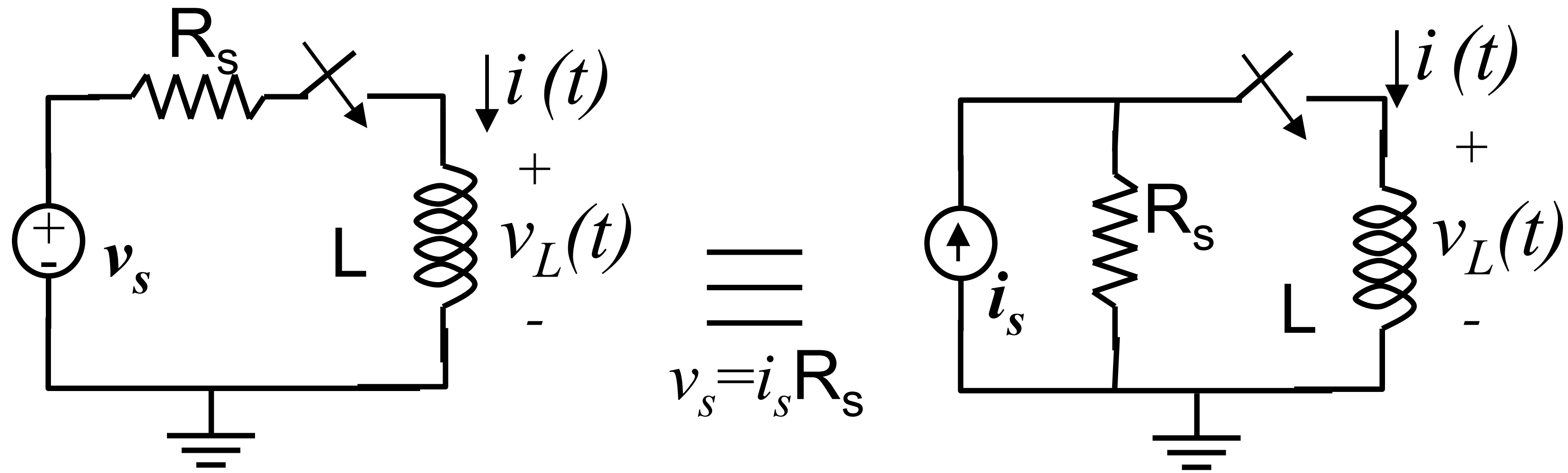
ECSE 210: Circuit Analysis

Lecture #6: RL/RC circuits

RL Circuits



KVL

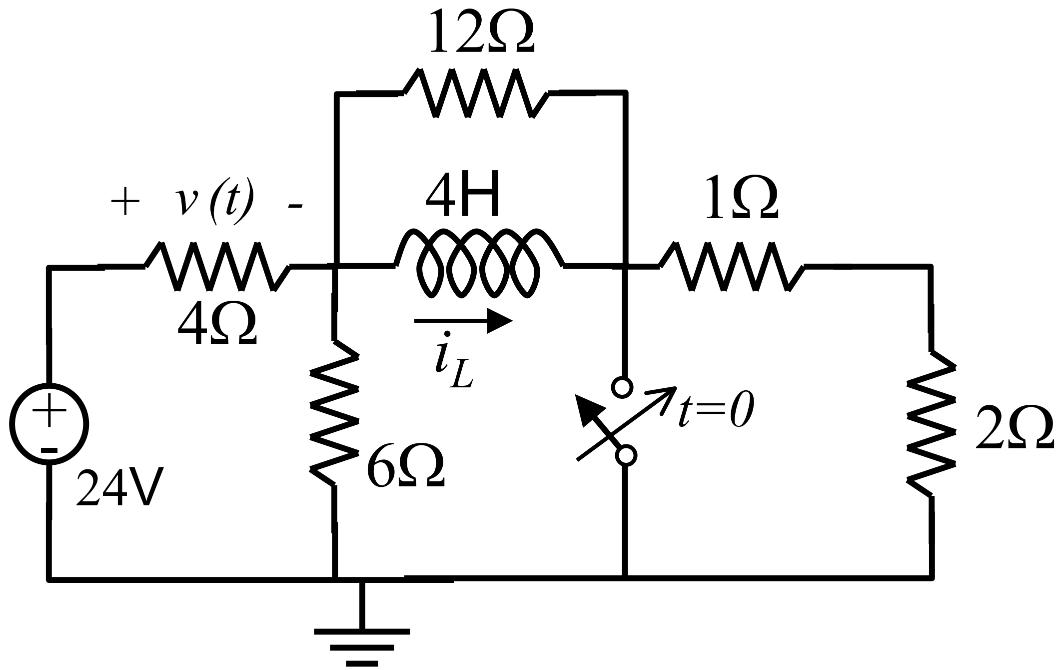
$$L\frac{di(t)}{dt} + R_s i(t) = v_s \quad \Rightarrow \quad \frac{di(t)}{dt} + \frac{R_s}{L} i(t) = \frac{v_s}{L}$$

$$i(t) = C_1 + C_2 e^{-\frac{t}{T_C}} \qquad T_C = \frac{L}{R_s}$$

C_1 and C_2 are determined from the boundary conditions

Example: RL Circuit

→ Find $v(t)$ for $t>0$



→ Follow six step procedure from Lecture 7

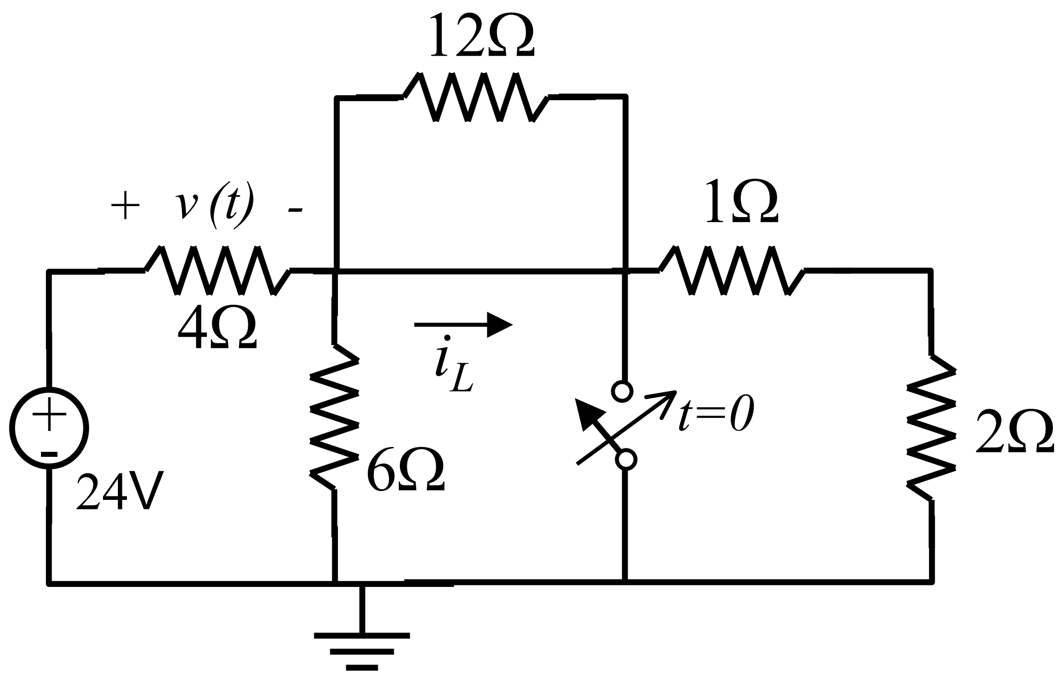
<u>Step 1:</u>

Assume solution of the form: $v(t) = C_1 + C_2 e^{-\frac{t}{T_C}}$

RL Circuits Example

Step 2:

Calculate the steady state inductor current at $i_L(0^-)$.
(Inductor is short-circuited, switch is still open)



→ 12Ω resistor is shorted out by the inductor.

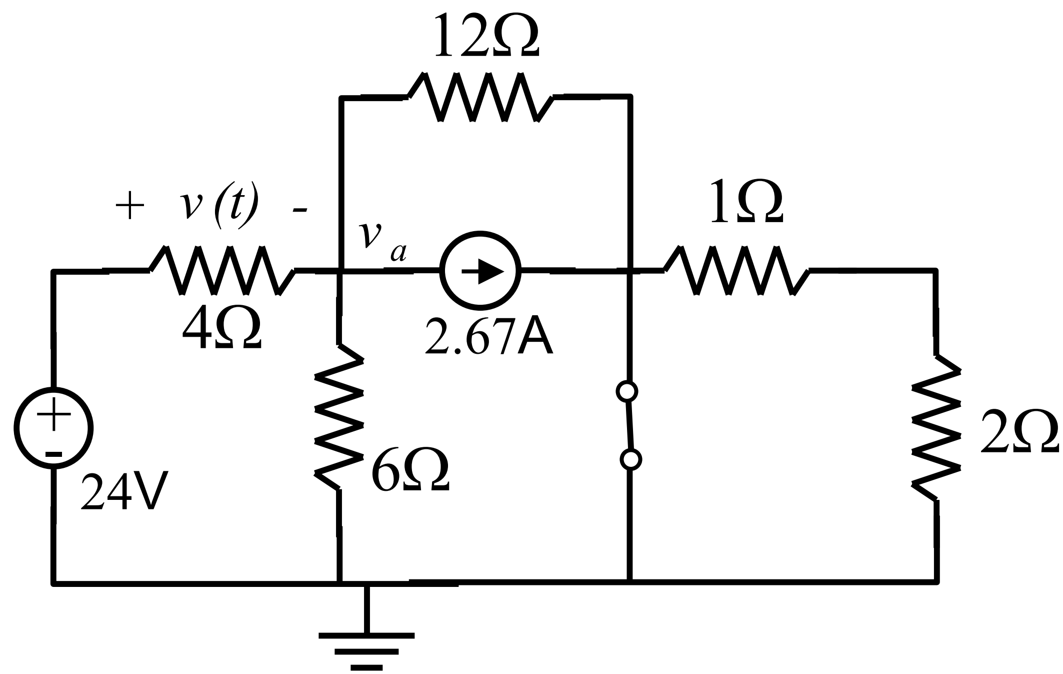
→ Current in 4Ω resistor is: $\dfrac{24V}{4\Omega + 6\Omega \,||\, (1\Omega + 2\Omega)} = 4A$

→ Using current divider, we get: $i_L(0^-) = \dfrac{6\Omega}{6\Omega + 3\Omega} 4A = 2.67A$

RL Circuits Example

Step 3:

Calculate $v(0^{+})$.
(Inductor is replaced by current source, switch is closed)



→ Define nodal voltage v_a at node a

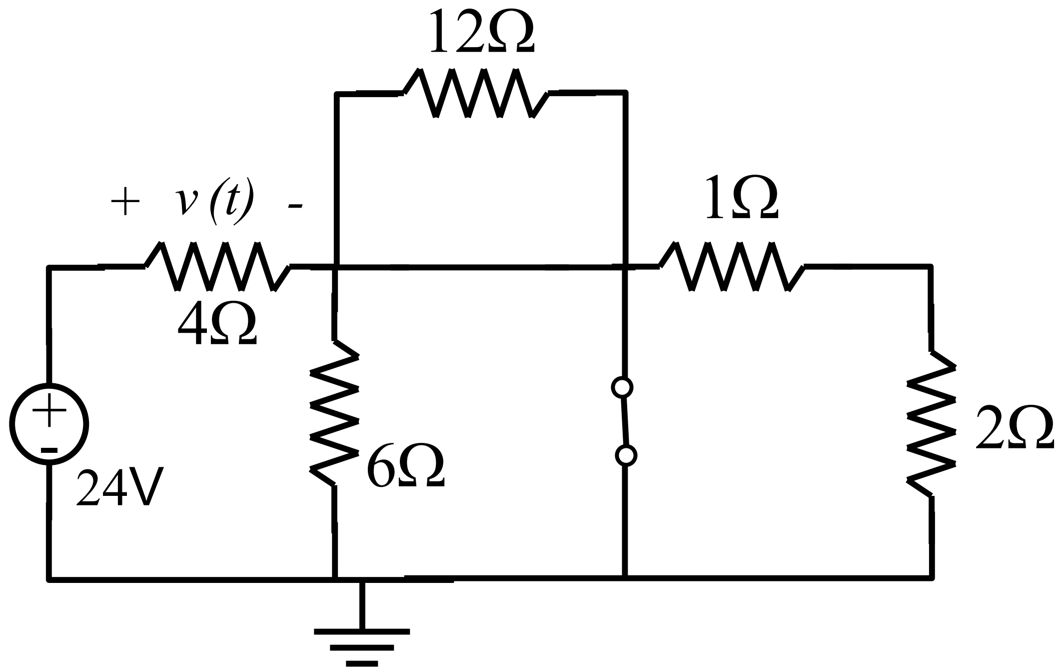
KCL at node a: $\frac{v_a - 24}{4\Omega} + \frac{v_a}{6\Omega \,||\, 12\Omega} + 2.67 = 0$ ⇒ $v_a = 6.67V$

⇒ $v(0^{+}) = 24 - 6.67 = 17.33V$

RL Circuits Example

Step 4:

Calculate $v(\infty)$.
(Inductor is replaced by short circuit, switch is closed)



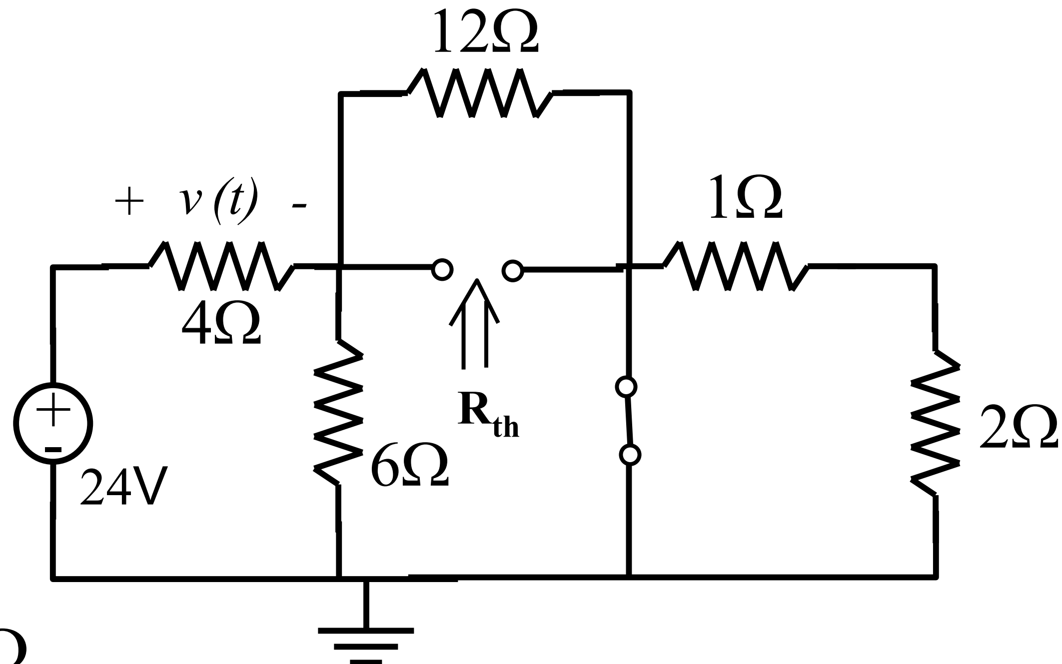
$$v(\infty) = 24V$$

Step 5:

Calculate constants: $C_1 = v(\infty) = 24V$

$$C_2 = v(0^+) - v(\infty) = -6.67V$$

RL Circuits Example



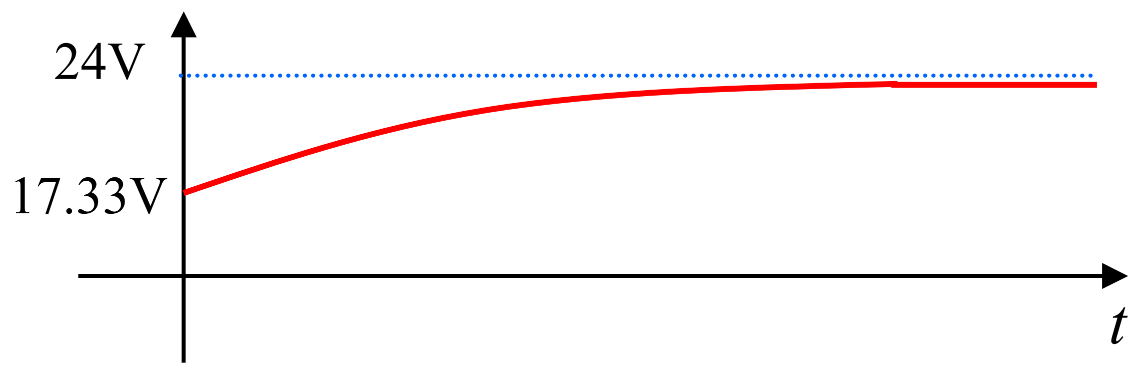
Step 6:

Calculate the time constant T_c

$$R_{th} = 4\Omega \,||\, 6\Omega \,||\, 12\Omega = 2\Omega$$

$$T_c = \frac{4H}{2\Omega} = 2s$$

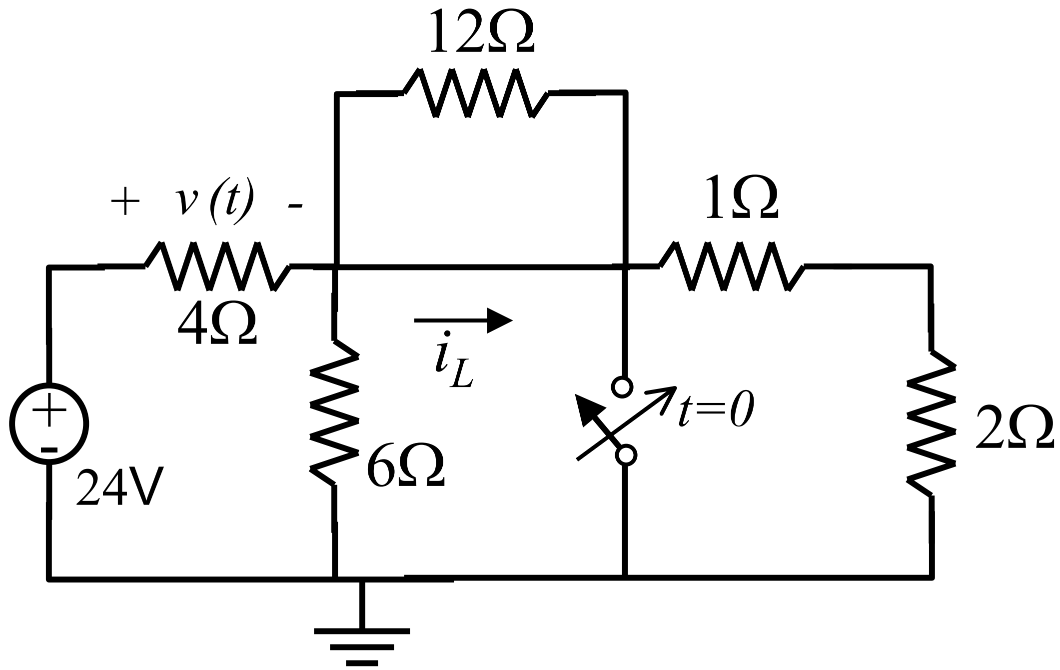
$$v(t) = 24 - 6.67e^{-\frac{t}{2}}$$



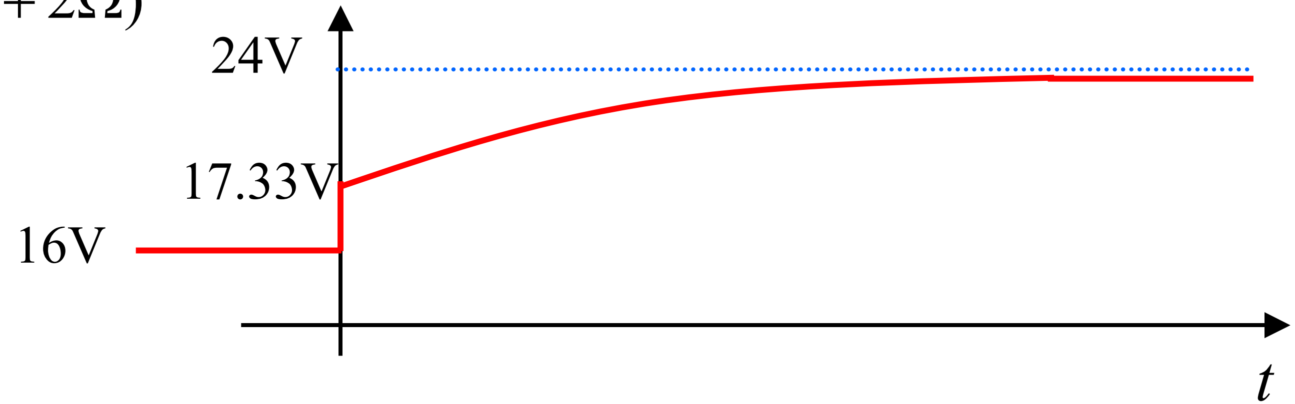
RL Circuits Example

What about $t<0$?

→ Switch is open, inductor replaced by short circuit

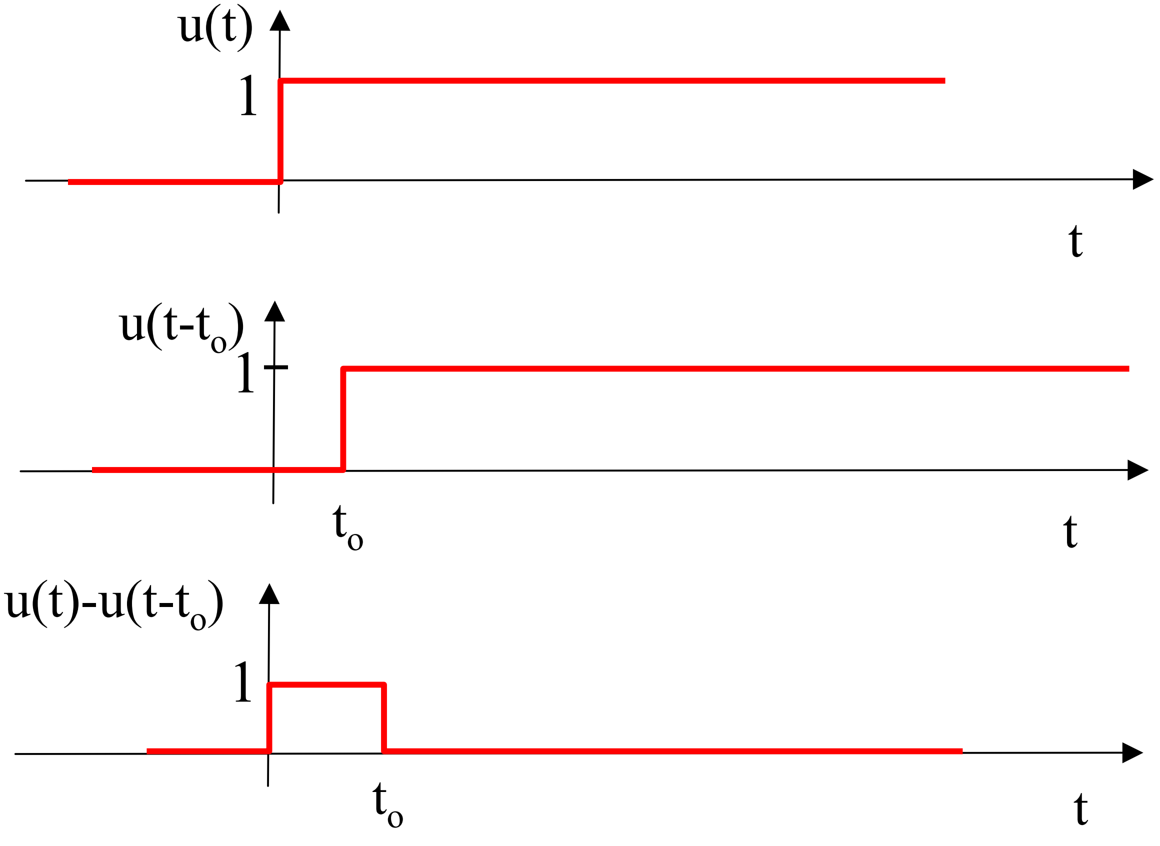


$$v(t) = \frac{4\Omega}{4\Omega + 6\Omega \,||\, (1\Omega + 2\Omega)} 24V = 16V \quad ; t < 0$$



Step Function

$$u(t) = \begin{cases} 0 & ;t < 0 \\ 1 & ;t > 0 \end{cases}$$

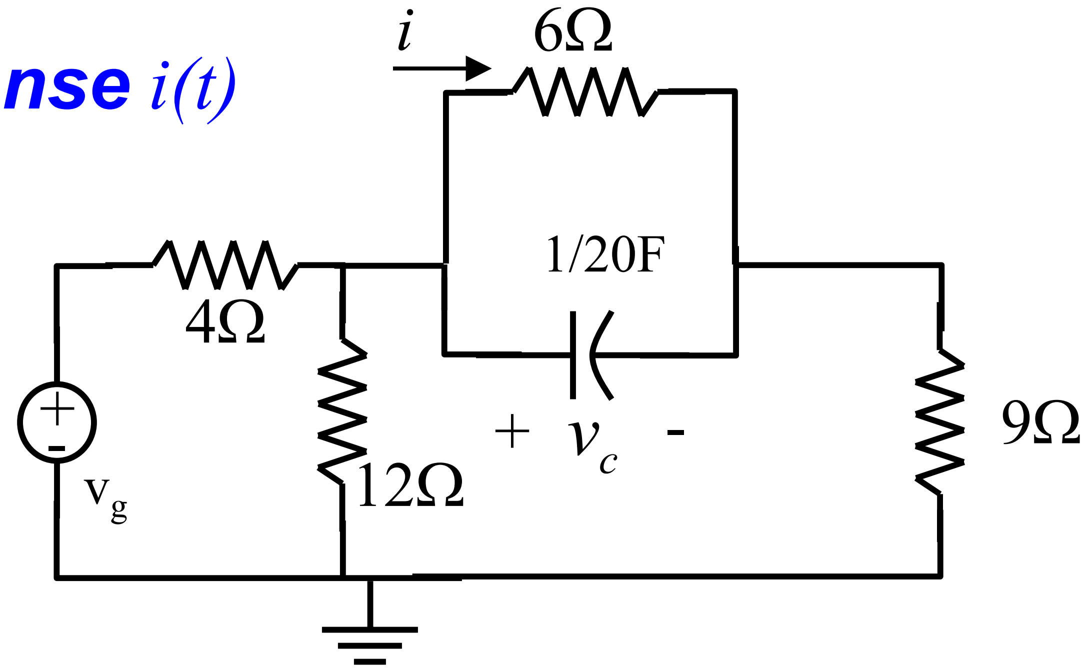


Example: Step Response

Find the ***step response*** $i(t)$

Step response

$v_g = u(t)$ Volts



Assume solution of the form: $i(t) = C_1 + C_2 e^{-\frac{t}{T_C}}$

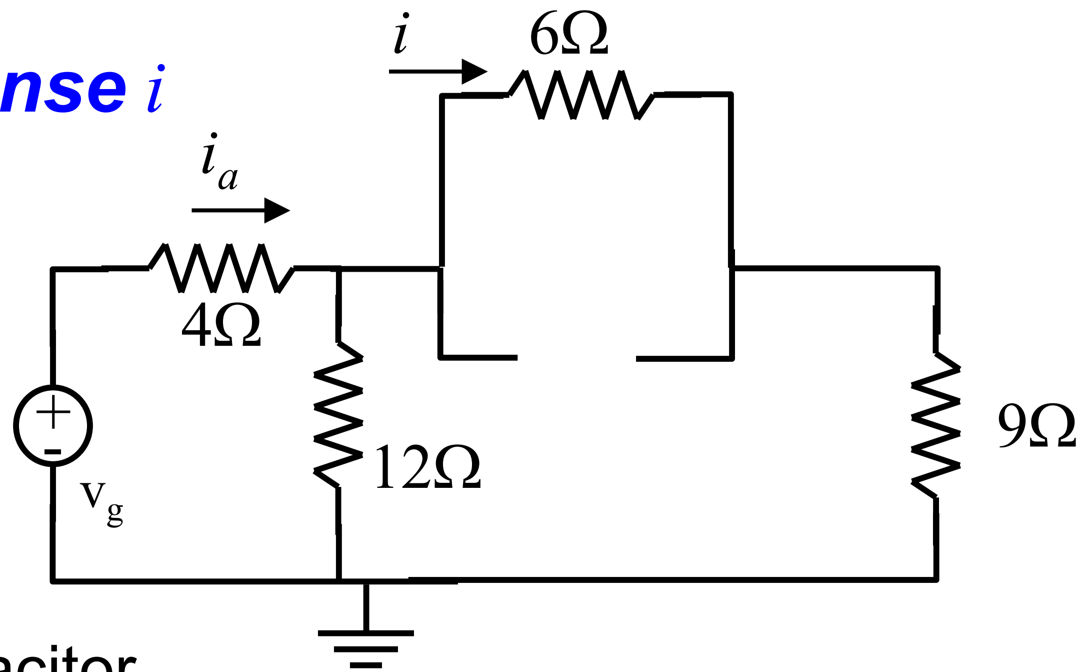
For t<0 the steady state capacitor voltage is $v_c = 0$

For t=0$^+$ Capacitor is replaced by a voltage source $= v_c = 0$V

→ $i(0^+) = 0$A

Example: Step Response

Find the ***step response*** i



At $t=\infty$ open circuit Capacitor

$$i_a = \frac{1V}{4\Omega + 12\Omega \,||\, (6\Omega + 9\Omega)} = \frac{1}{10.67} = 93.7mA$$

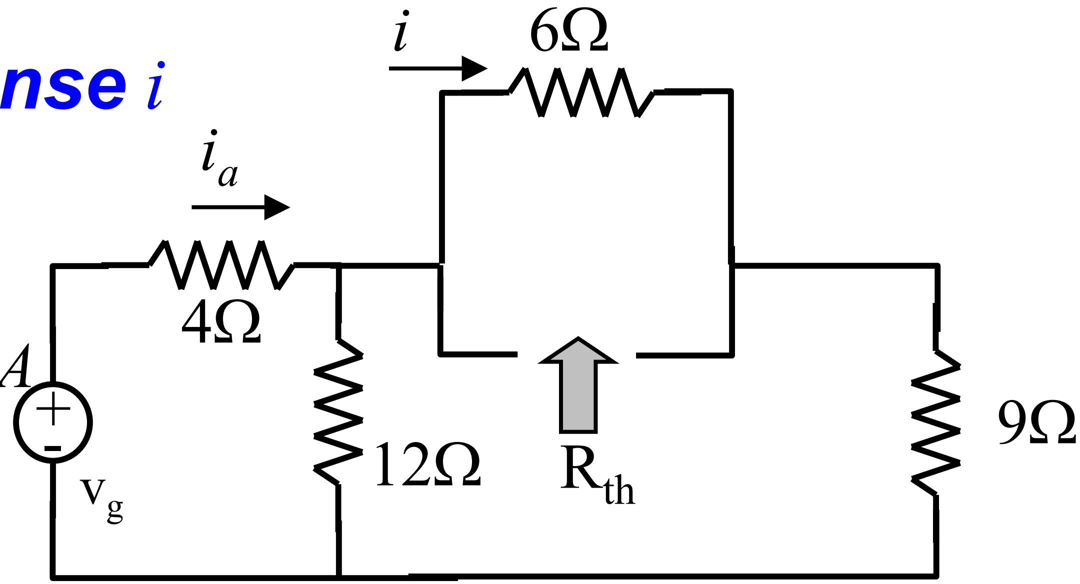
$$i(\infty) = \frac{12\Omega}{12\Omega + 15\Omega} 93.7mA = 41.7mA$$

Example: Step Response

Find the ***step response*** i

$$C_1 = i(\infty) = 41.7mA$$

$$C_2 = i(0^+) - i(\infty) = -41.7mA$$



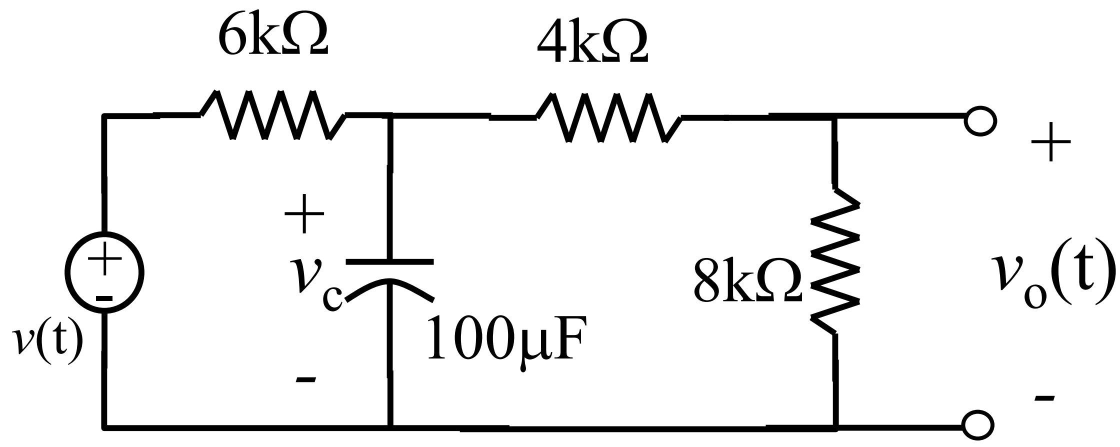
$$R_{th} = (4\Omega \,||\, 12\Omega + 9\Omega) \,||\, 6\Omega = 4\Omega$$

$$T_C = R_{th}C = (4\Omega)(1/20F) = 0.2s$$

The step response is: $i(t) = (41.7 - 41.7e^{-\frac{t}{0.2}})mA$

Example with an Input Pulse

Find the response $v_o(t)$



<u>For t<0</u>

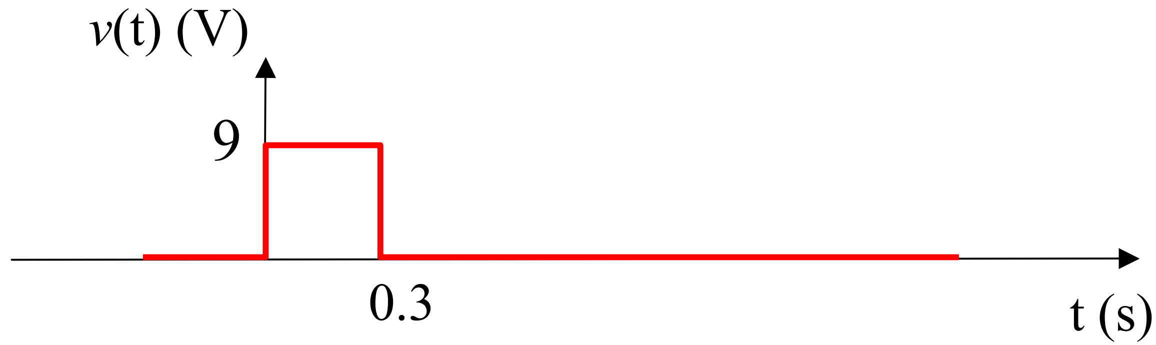
$$v_o(t) = 0V$$

$$v_c(0^-) = 0V$$

<u>For t=0+</u>

Replace capacitor with
Voltage source = 0V

$$v_o(0^+) = 0V$$

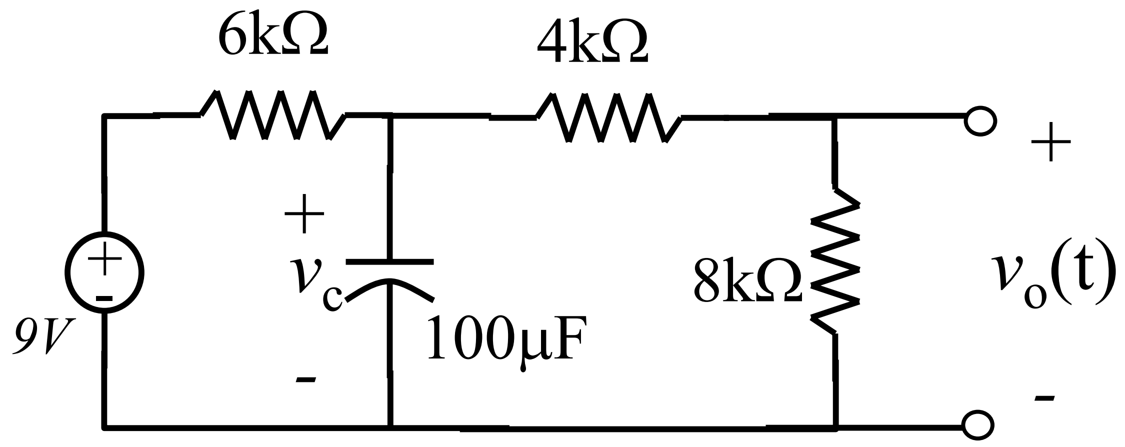


Example with an Input Pulse

Find the response $v_o(t)$

<ins>For t=∞</ins>

Replace capacitor with open circuit



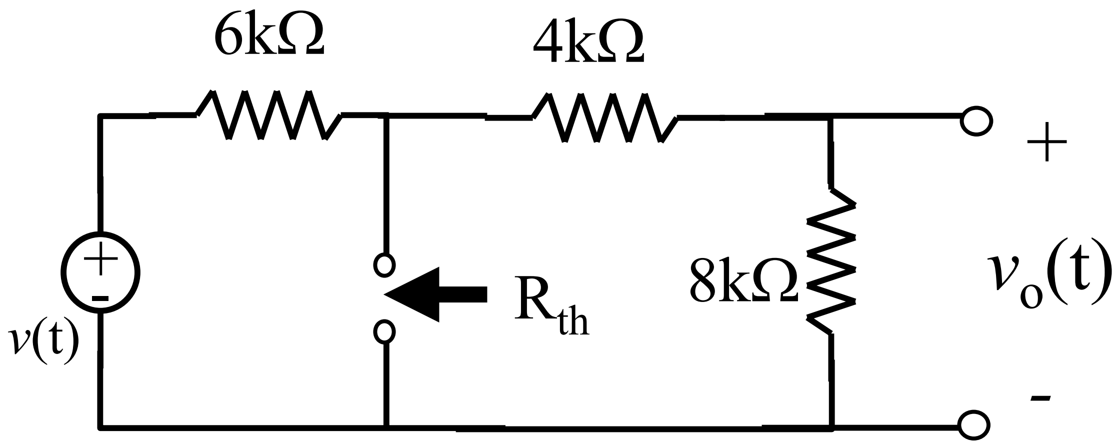
$$v_o(\infty) = \frac{8k\Omega}{6k\Omega + 4k\Omega + 8k\Omega} 9V = 4V$$

$$C_1 = v(\infty) = 4V$$

$$C_2 = v(0^+) - v(\infty) = -4V$$

Example with an Input Pulse

Find the response $v_o(t)$



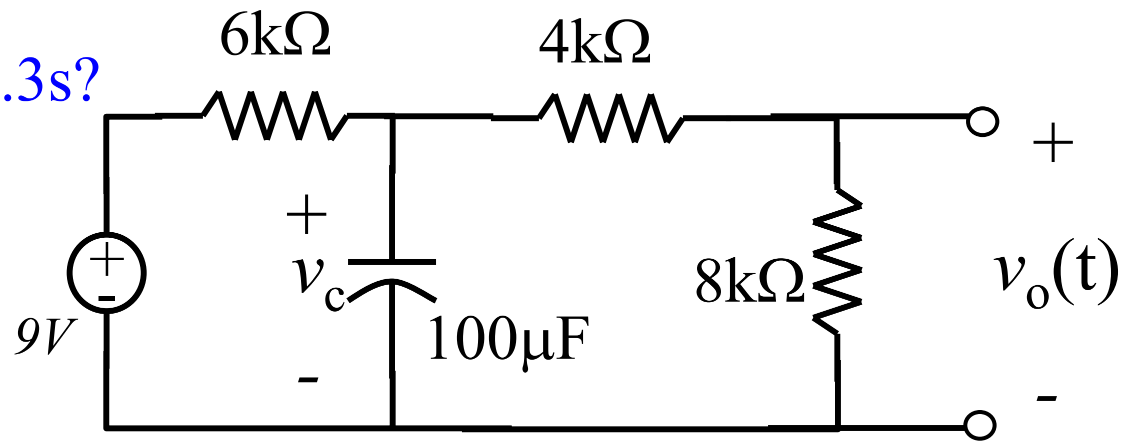
$$R_{th} = 6k\Omega \,||\, (4k\Omega + 8k\Omega) = 4k\Omega$$

$$T_C = (4k\Omega)(100\mu F) = 0.4s$$

$$v_o(t) = 4 - 4e^{-\frac{t}{0.4}}V \qquad ; 0 < t < 0.3s$$

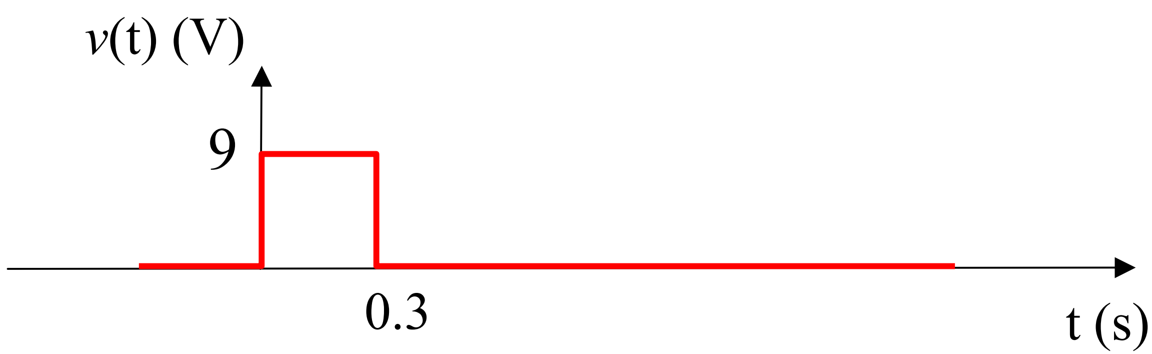
Example with an Input Pulse

→What happens before t=0.3s?



For t=0.3-

$$v_o(0.3^-) = 4 - 4e^{-\frac{0.3}{0.4}} = 2.11V$$



By voltage divider:

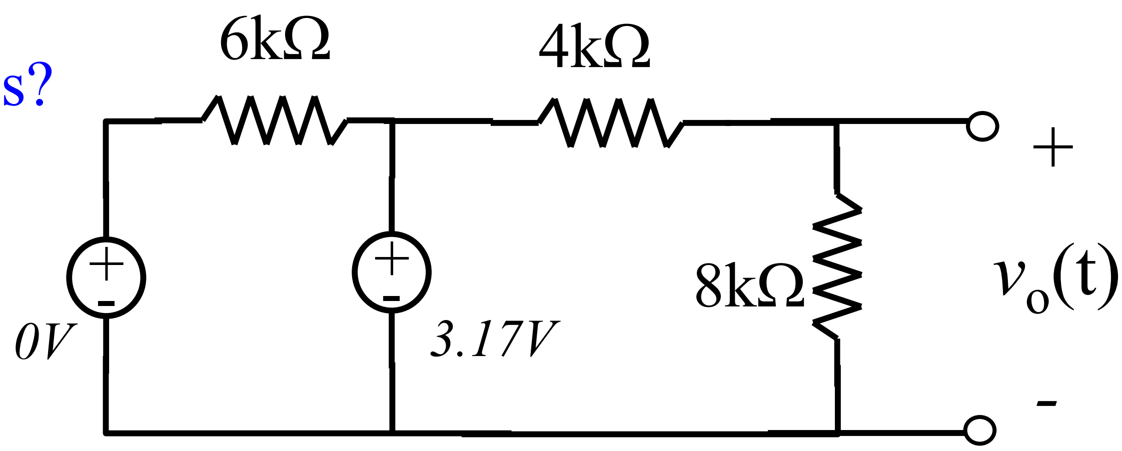
$$v_o = \frac{8}{4+8} v_c$$

$$v_c = \frac{3}{2} v_o$$

$$v_c(0.3^-) = \frac{3}{2} 2.11V = 3.17V$$

Example with an Input Pulse

→What happens after t=0.3s?



For t=0.3+

Replace capacitor with voltage source:

$$v_o(0.3^+) = \frac{8}{4+8} 3.17 = 2.11V$$

For t=∞

Replace capacitor with open circuit:

$$v_o(\infty) = 0V$$

Example with an Input Pulse

→Assume solution:

$$v_o(t) = C_1 + C_2 e^{-\frac{t-0.3}{T_C}} \qquad ;t > 0.3s$$

$$v_o(\infty) = C_1 = 0V$$

$$v_o(0.3^+) = C_1 + C_2 = C_2 = 2.11V$$

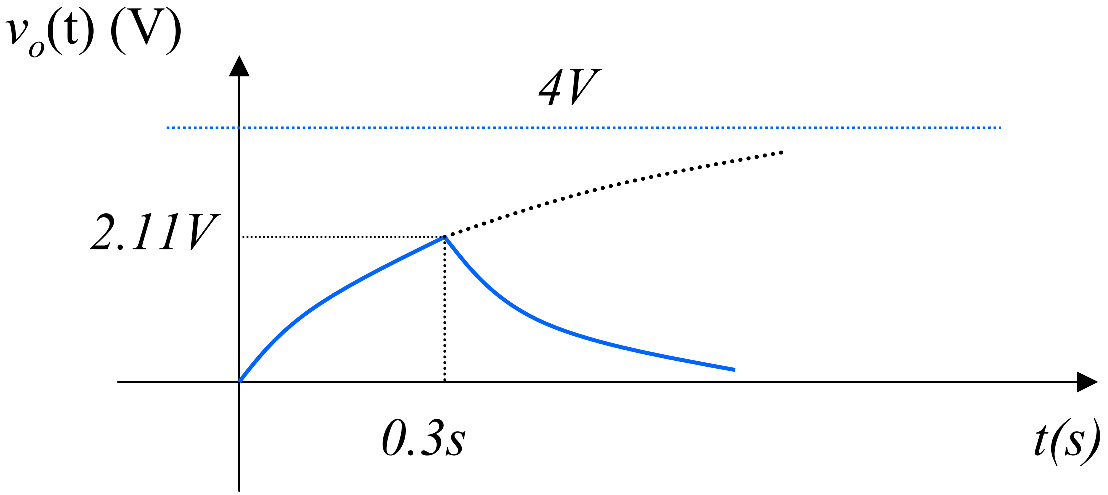
$$R_{th} = 6k\Omega \,||\, (4k\Omega + 8k\Omega) = 4k\Omega$$

$$T_C = (4k\Omega)(100\mu F) = 0.4s$$

$$v_o(t) = 2.11e^{-\frac{t-0.3}{0.4}} \qquad ;t > 0.3s$$

Example with an Input Pulse

$$v_o(t) = \begin{cases} 0 & ;t < 0 \\ 4 - 4e^{-\frac{t}{0.4}} V & ;0 < t < 0.3s \\ 2.11e^{-\frac{t-0.3}{0.4}} V & ;t > 0.3s \end{cases}$$



Exponential Input

So far we have looked at:

$$\frac{dx(t)}{dt} + ax(t) = A \quad \Rightarrow \quad x(t) = C_1 + C_2 e^{-at}$$

$C_2 e^{-at}$ solves the *homogeneous equation*

→ **Called the *natural response***

C_1 is a *particular solution* to the ODE

→ **Called the forced response**

→ C_1 and C_2 are determined from the boundary conditions.

Exponential Input

Now consider:

$$\frac{dx(t)}{dt} + ax(t) = A_o e^{-bt} \quad \Rightarrow \quad x(t) = C_1 e^{-bt} + C_2 e^{-at}$$

$C_2 e^{-at}$ solves the *homogeneous equation*

→ **Called the *natural response***

$C_1 e^{-bt}$ is a *particular solution* to the ODE

→ **Called the *forced response***

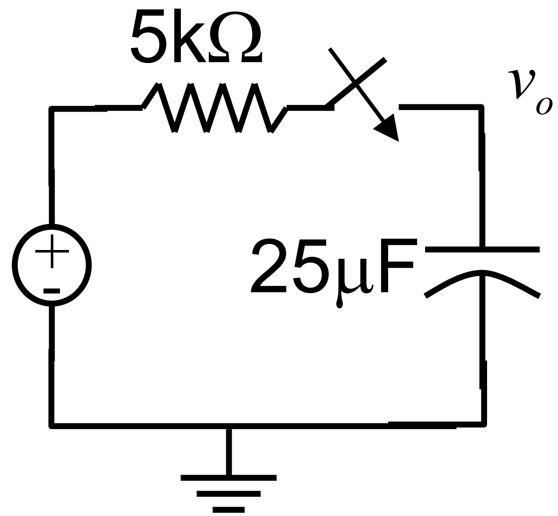
Example: Exponential Input

Assume:

$$v_o(0^-) = -18\,V$$

$$v_s(t) = 6e^{-2t}u(t) \quad V$$

$$v_s(t) = 6e^{-2t}u(t) \quad V$$



KCL at top right node:

$$\frac{v_o(t) - v_s(t)}{R} + C\frac{dv_o(t)}{dt} = 0 \qquad ;t > 0$$

$$\frac{dv_o(t)}{dt} + 8v_o(t) = 48e^{-2t}$$

$$\Rightarrow \quad v_o(t) = C_1e^{-2t} + C_2e^{-\frac{t}{T_C}}$$

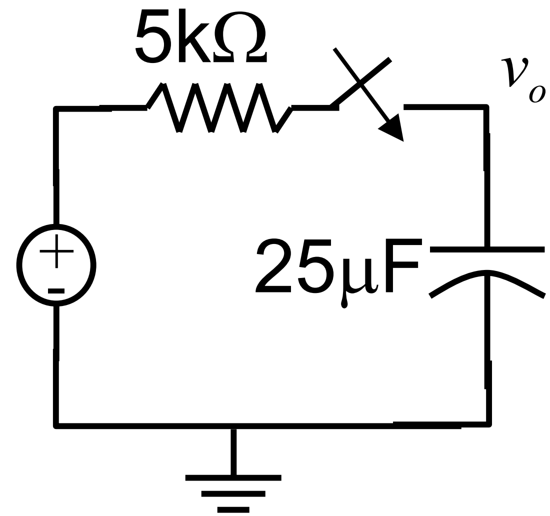
Example:Exponential Input

$$\frac{dv_o(t)}{dt} + 8v_o(t) = 48e^{-2t}$$

$$T_C = (5k\Omega)(25\mu F) = \frac{1}{8}s$$

$$v_o(t) = C_1 e^{-2t} + C_2 e^{-8t}$$

$v_s(t) = 6e^{-2t}V$



Substitute v_o into ODE

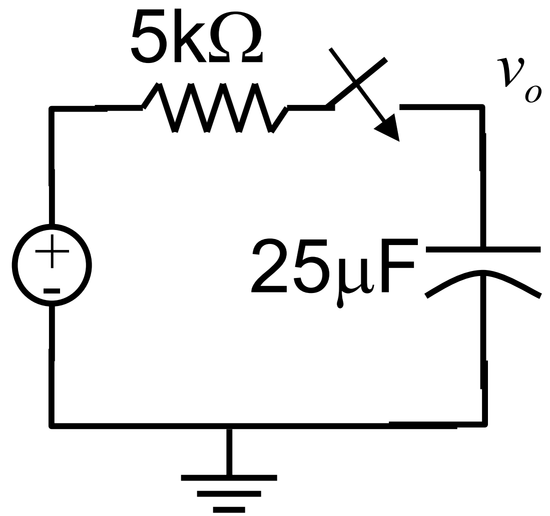
$$-2C_1 e^{-2t} - 8C_2 e^{-8t} + 8C_1 e^{-2t} + 8C_2 e^{-8t} = 48e^{-2t}$$

Equate coefficients of e^{-2t}

$$\Rightarrow C_1 = 8$$

Example: Exponential Input

$$v_o(0^+) = -18V = C_1 + C_2$$



$$C_2 = -26$$

$$v_o(t) = 8e^{-2t} - 26e^{-8t}V \qquad ;t > 0$$