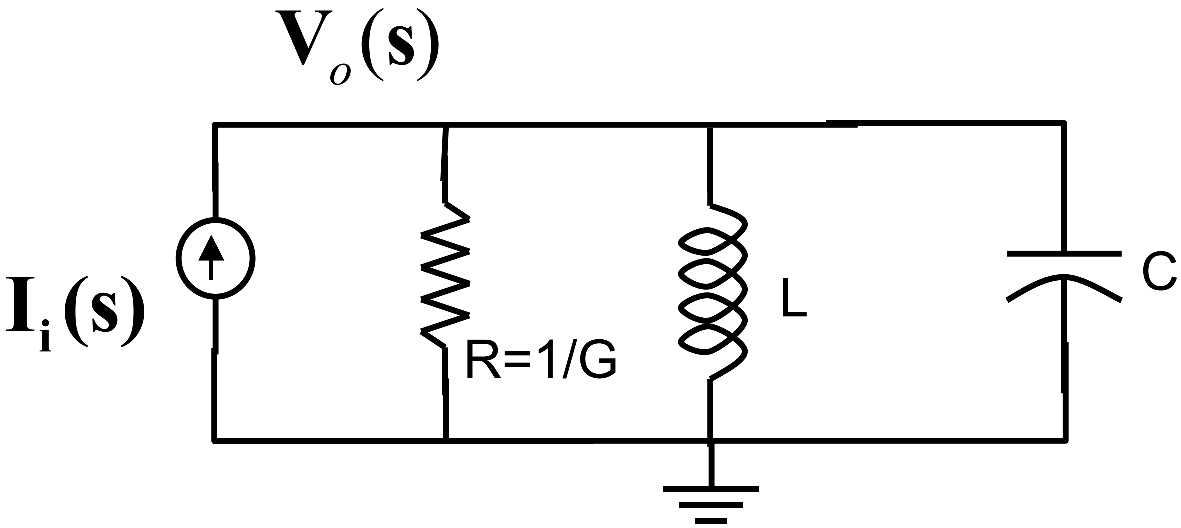
ECSE 210: Circuit Analysis

Lecture #27:

Resonant Circuits

Parallel Resonant Circuit



$$\mathbf{H(s)} = \frac{\mathbf{V_o(s)}}{\mathbf{I_i(s)}} = \mathbf{Z(s)} = \frac{1}{\frac{1}{R} + \frac{1}{\mathbf{s}L} + \mathbf{s}C} = \frac{1}{G + \frac{1}{\mathbf{s}L} + \mathbf{s}C}$$

$$\mathbf{H(s)} = \frac{\mathbf{V_o(s)}}{\mathbf{I_i(s)}} = \mathbf{Z(s)} = \frac{\frac{1}{C}\mathbf{s}}{\mathbf{s}^2 + \frac{G}{C}\mathbf{s} + \frac{1}{LC}}$$

Parallel Resonant Circuit

$$\mathbf{H}(s) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{I_i}(\mathbf{s})} = \mathbf{Z}(\mathbf{s}) = \frac{\frac{1}{C}\mathbf{s}}{\mathbf{s}^2 + \frac{G}{L}\mathbf{s} + \frac{1}{LC}}$$

$$\omega_o = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{G/C}{2\omega_o}$$

or $$\zeta = \frac{G/C}{2\omega_o} = \frac{G}{C}\frac{1}{2}\sqrt{LC} = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Parallel Resonant Circuit

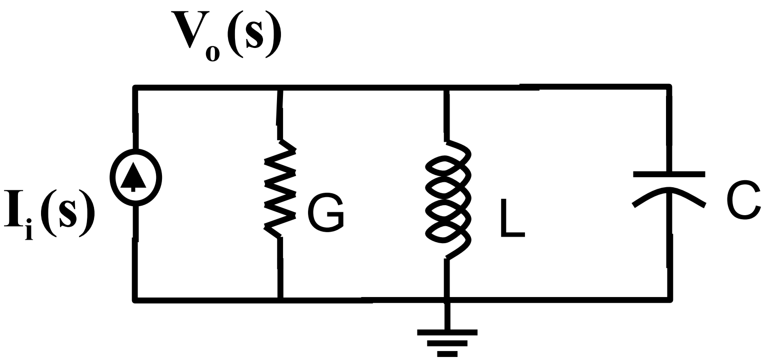
Also

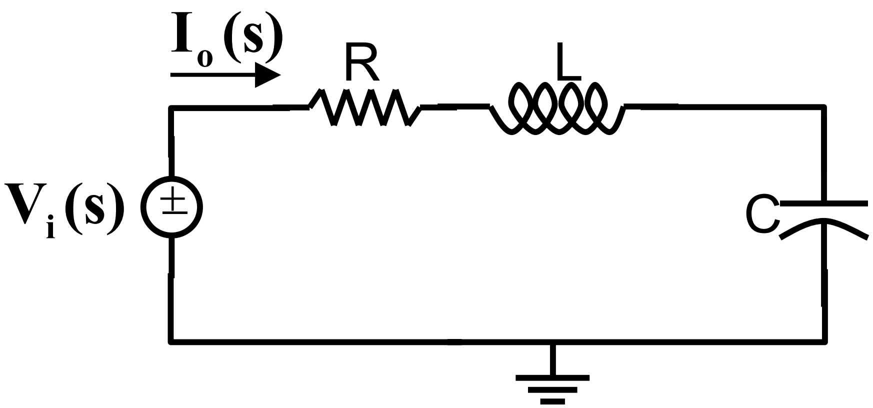
Z(**s**) is real when:

$$\omega_r = \omega_o = \sqrt{\frac{1}{LC}}$$ ⇨ Resonant frequency

$$Q_o = \omega_o RC = \frac{R}{L\omega_o} = R\sqrt{\frac{C}{L}}$$

Series/Parallel Resonant Circuit





$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V_o}(\mathbf{s})}{\mathbf{I_I}(\mathbf{s})} = \mathbf{Z}(\mathbf{s}) = \frac{\frac{1}{C}\mathbf{s}}{\mathbf{s}^2 + \frac{G}{C}\mathbf{s} + \frac{1}{LC}}$$

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{I}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \mathbf{Y}(s) = \frac{\frac{1}{L}\mathbf{s}}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}}$$

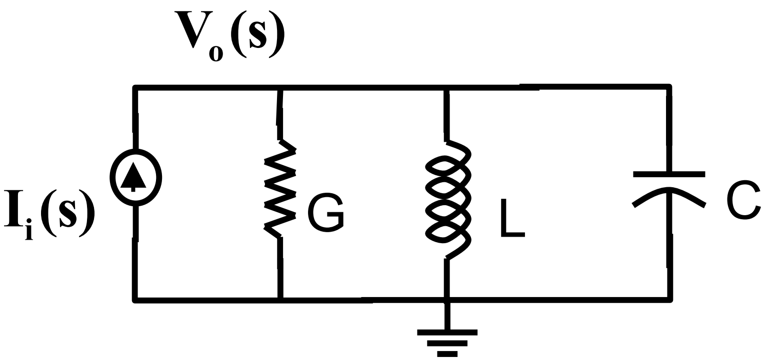
$$\zeta = \frac{G}{2}\sqrt{\frac{L}{C}} = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

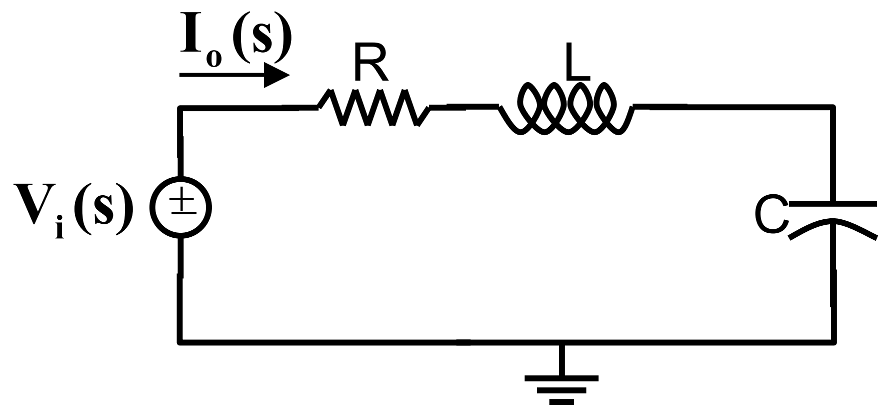
$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$Q_o = \omega_o RC = \frac{R}{L\omega_o} = R\sqrt{\frac{C}{L}}$$

$$Q_o = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Series/Parallel Resonant Circuit





$$\omega_o = \frac{1}{\sqrt{LC}} \rightarrow$$ Resonant frequency

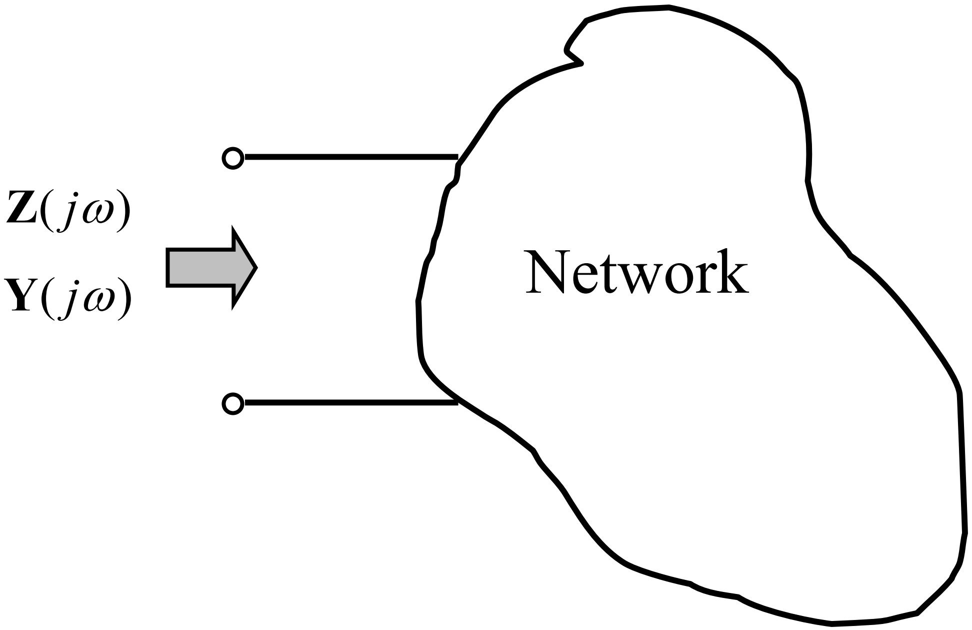
$$BW = \frac{\omega_o}{Q_o}$$

$$\omega_{1,2} = \omega_o\left[\sqrt{\left(\frac{1}{2Q_o}\right)^2 + 1} \pm \frac{-1}{2Q_o}\right]$$

$$\omega_o = \sqrt{\omega_1 \omega_2}$$

Resonance

We consider the circuit to be in resonance when the admittance or impedance is real. The frequency at which this occurs is the resonant frequency.

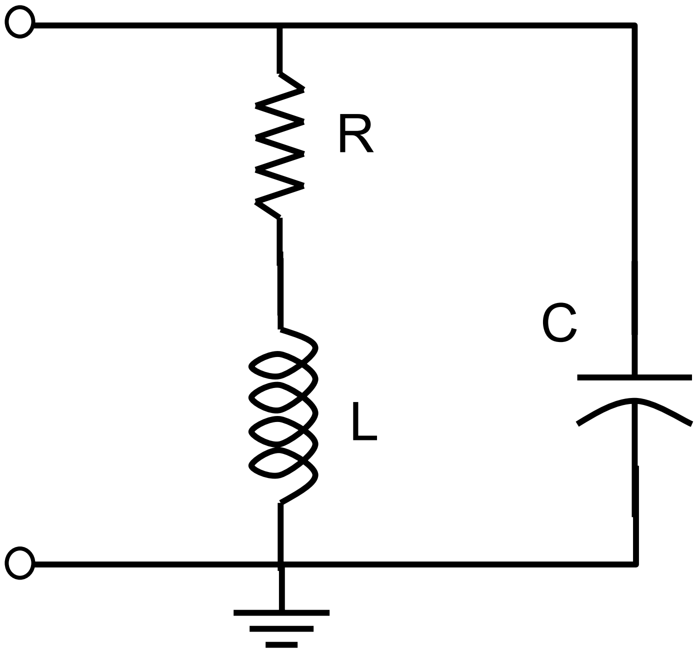


Example

Find the resonant frequency of the following circuit:

$$\mathbf{Y}(j\omega) = j\omega C + \frac{1}{R + j\omega L} = j\omega C + \frac{R - j\omega L}{R^2 + \omega^2 L^2}$$

$$\mathbf{Y}(j\omega) = \frac{R}{R^2 + \omega^2 L^2} + j\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)$$



Circuit at resonance when Y is real.
Therefore:

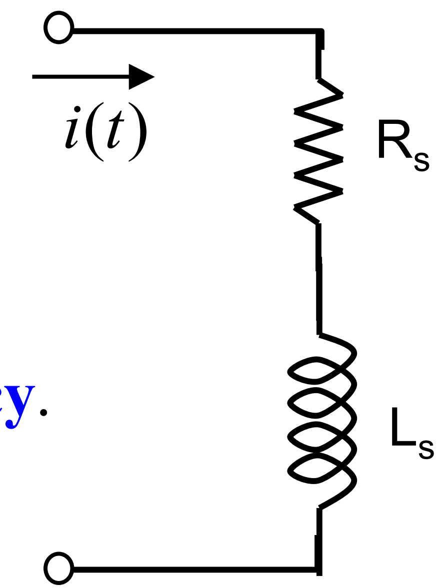
$$\omega C - \frac{\omega L}{R^2 + \omega^2 L^2} = 0$$

$$\Rightarrow \quad \omega_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Practical Inductors

In practice, **real** components have more complex models. For example, an **inductor coil** would have a resistance associated with it due to the non-ideal conductor. In this case, losses are associated with the inductor, and a quality factor can be defined.

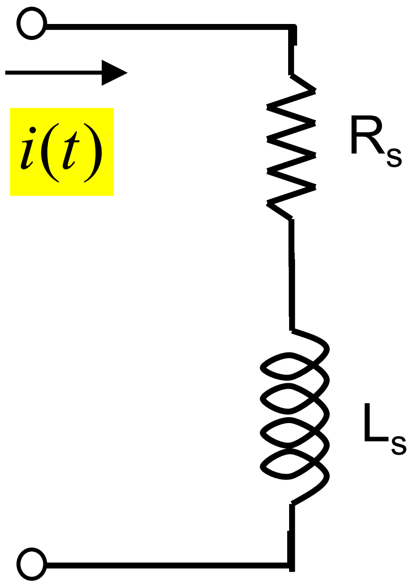
$$Q = 2\pi \frac{\text{maximum total energy stored}}{\text{energy dissipated per cycle}}$$



The quality factor is a **function of frequency**.

Practical Inductors

$i(t) = I_m \cos(\omega t)$



Maximum energy stored: $E_L = \frac{1}{2} L_s I_m^2$

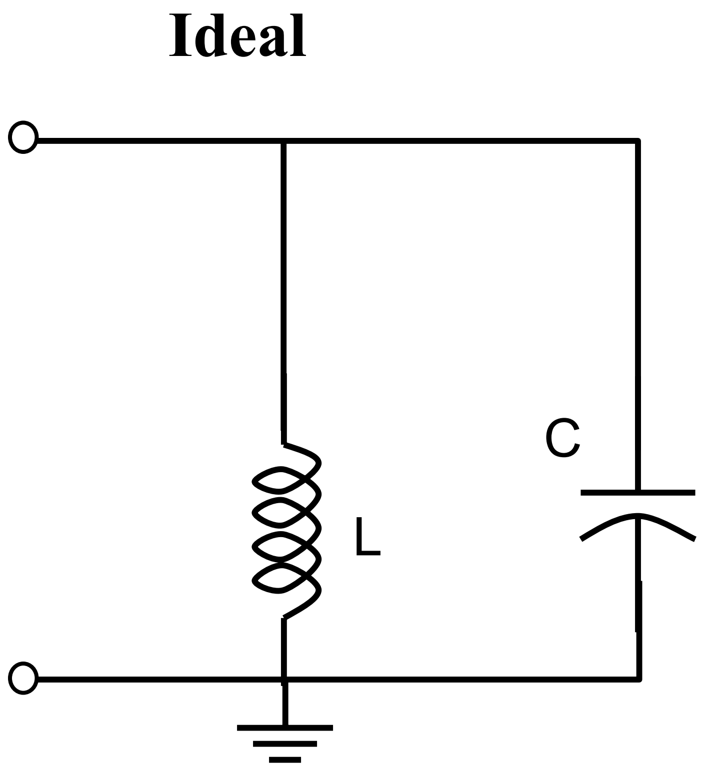
Average power dissipated in resistor:

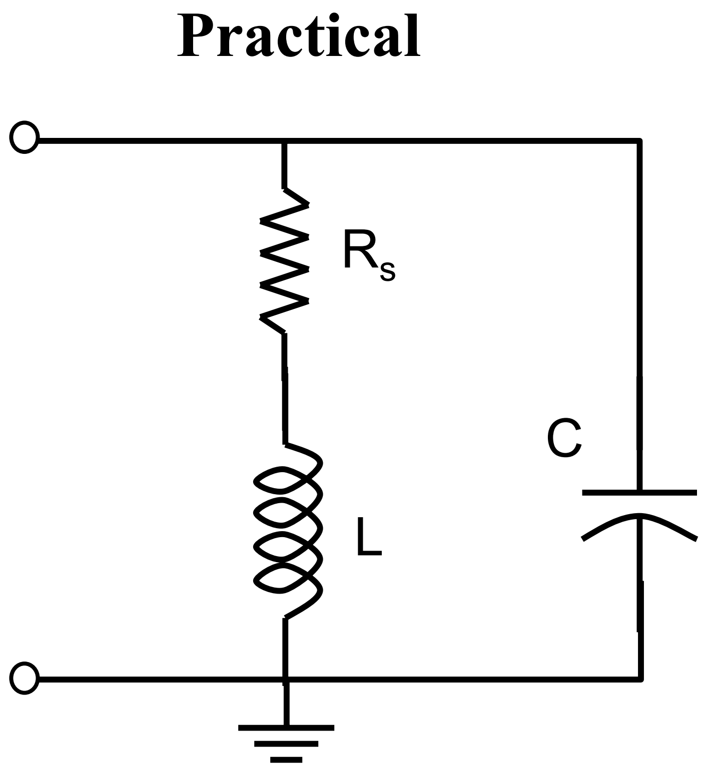
$$P_R = \frac{1}{2} R_s I_m^2$$

Energy dissipated per cycle: $E_R = TP_R = \left(\frac{2\pi}{\omega}\right)\left(\frac{1}{2} R I_m^2\right)$

$$Q = 2\pi \frac{\frac{1}{2} L_s I_m^2}{\left(\frac{2\pi}{\omega}\right)\left(\frac{1}{2} R_s I_m^2\right)} = \frac{\omega L_s}{R_s} = \frac{X_s}{R_s}$$ $\Leftarrow$ **function of frequency**

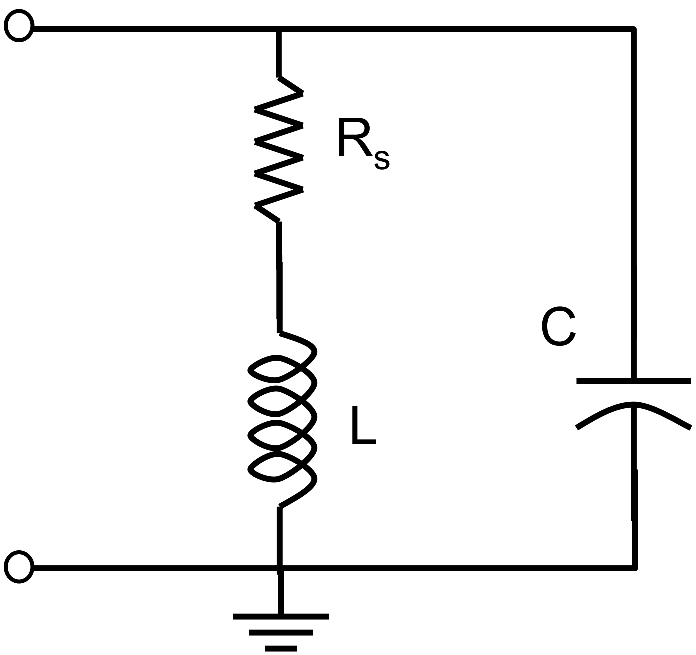
Practical Tuned Circuits



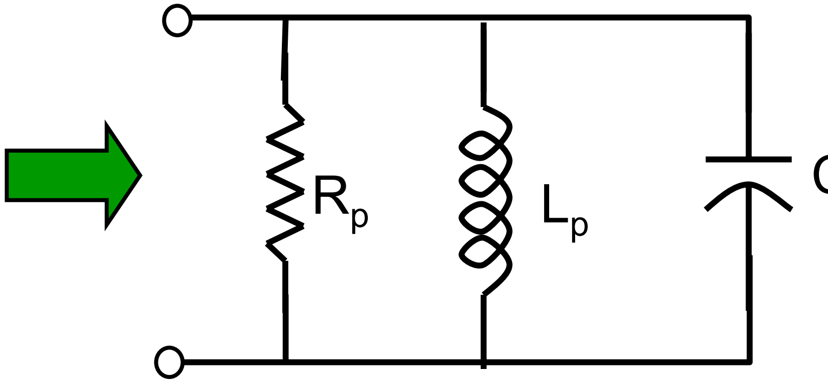


Practical Tuned Circuits

Practical



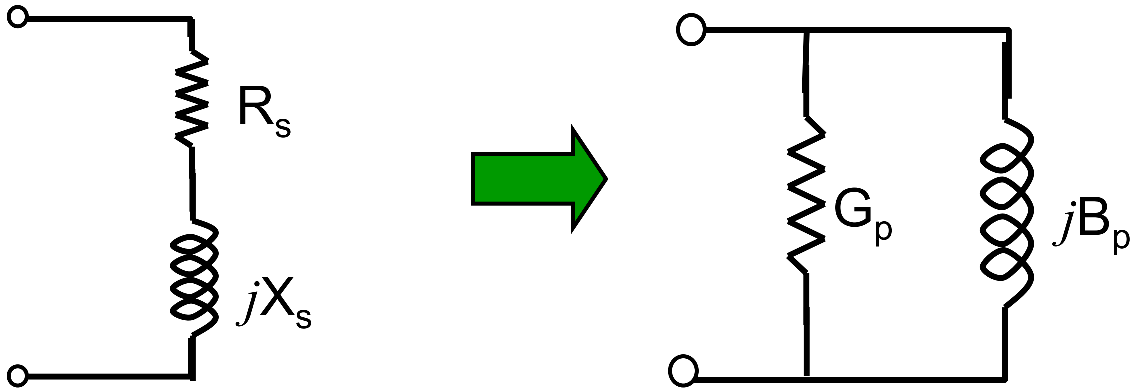
Convenient



$$\mathbf{Y}(j\omega) = \frac{R}{R^2 + \omega^2 L^2} + j\left(\omega C - \frac{\omega L}{R^2 + \omega^2 L^2}\right)$$

$$= R_p + j\omega C + \frac{1}{j\omega L_p}$$

Practical Inductors

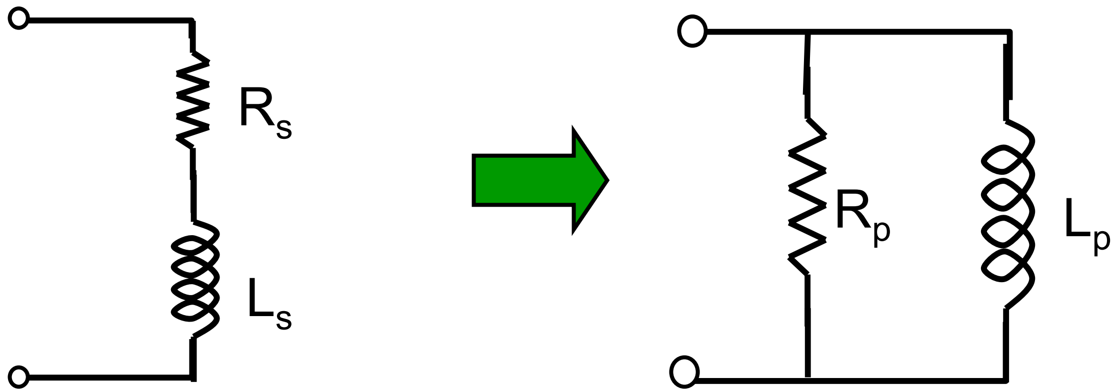


Consider the above circuits. Find the conductance G_p and susceptance B_p in terms of R_s and X_s so that the circuits have the same impedance at a frequency ω.

$$Y(j\omega) = G_p + jB_p = \frac{1}{Z(j\omega)} = \frac{1}{R_s + jX_s} = \frac{R_s - jX_s}{R_s^2 + X_s^2}$$

$$G_p = \frac{R_s}{R_s^2 + X_s^2} \qquad B_p = \frac{-X_s}{R_s^2 + X_s^2}$$

Practical Inductors



Consider the above circuits. Find R_p and L_p in terms of R_s and L_s at a frequency ω.

$$B_p = -\frac{1}{\omega L_p} = \frac{-X_s}{R_s^2 + X_s^2} = \frac{-\omega L_s}{R_s^2 + (\omega L_s)^2}$$

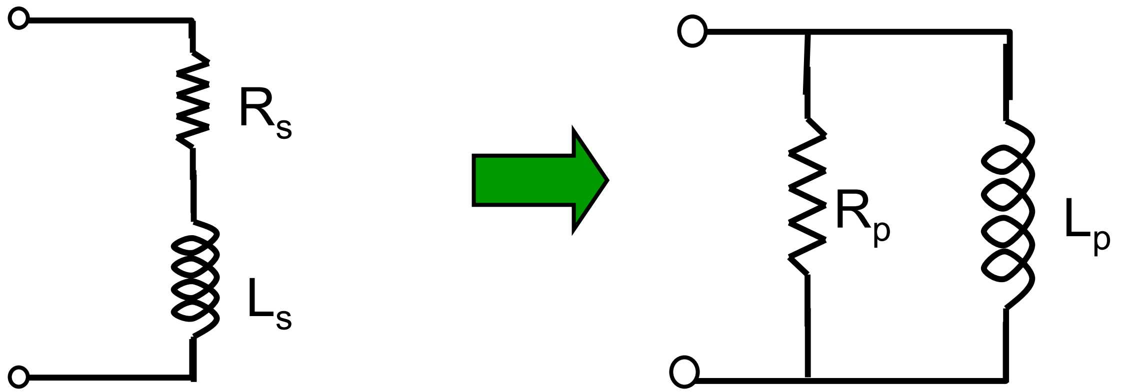
$$G_p = \frac{R_s}{R_s^2 + X_s^2}$$

$$L_p = \frac{R_s^2 + (\omega L_s)^2}{\omega^2 L_s} = \frac{R_s^2}{\omega^2 L_s} + L_s = L_s\left(1 + \frac{1}{Q^2}\right)$$

$$Q = \frac{\omega L_s}{R_s} = \frac{X_s}{R_s}$$

$$\boxed{L_p \cong L_s \quad \text{for} \quad Q > 10}$$

Practical Inductors



Consider the above circuits. Find R_p and L_p in terms of R_s and L_s at a frequency ω.

$$G_p = \frac{1}{R_p} = \frac{R_s}{R_s^2 + X_s^2}$$

$$Q = \frac{\omega L_s}{R_s} = \frac{X_s}{R_s}$$

$$R_p = R_s(1+Q^2)$$

$$R_p \cong R_s Q^2 = Q\omega L_s \quad \text{for} \quad Q > 10$$