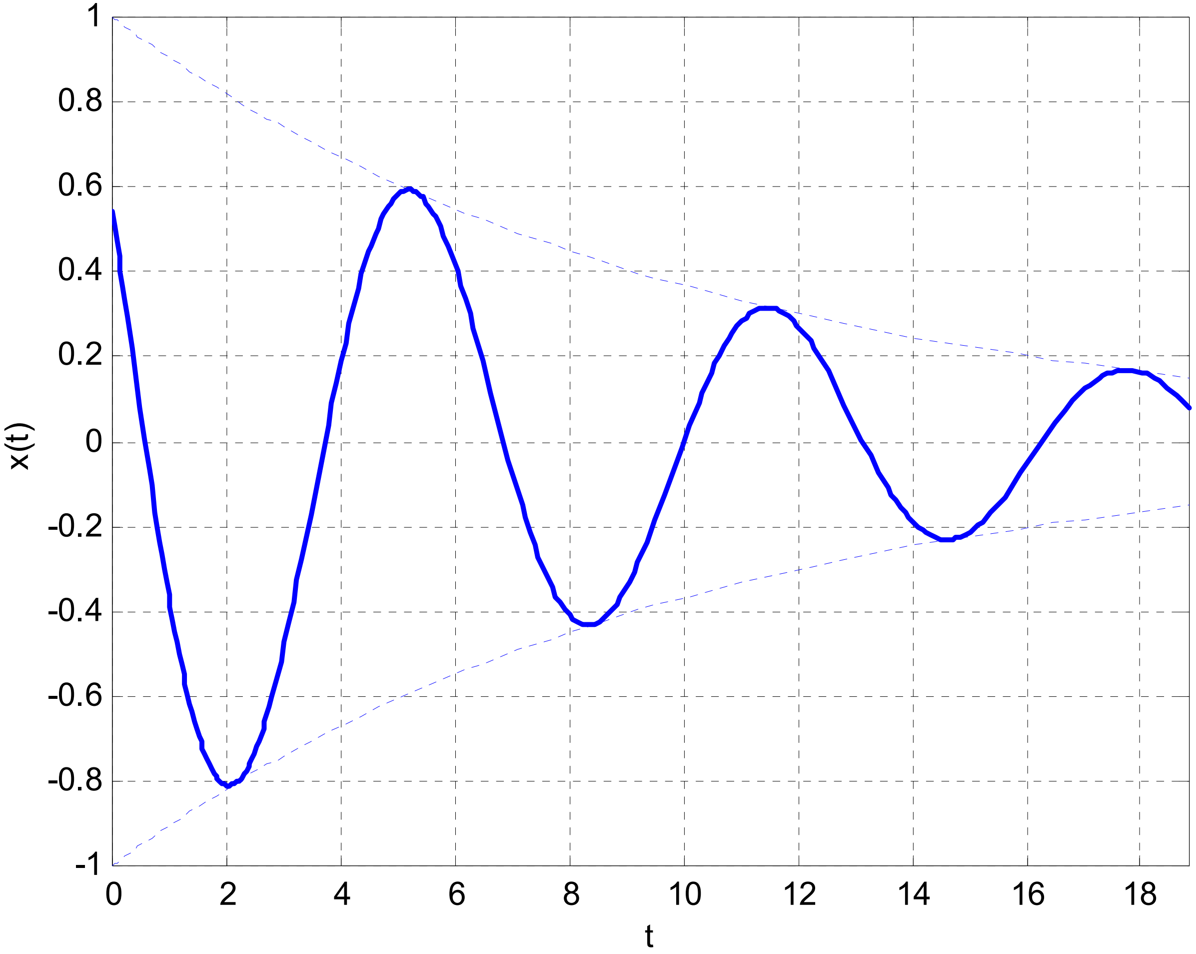
ECSE 210: Circuit Analysis

Lecture #25:

Generalized Phasors

Damped Sinusoids

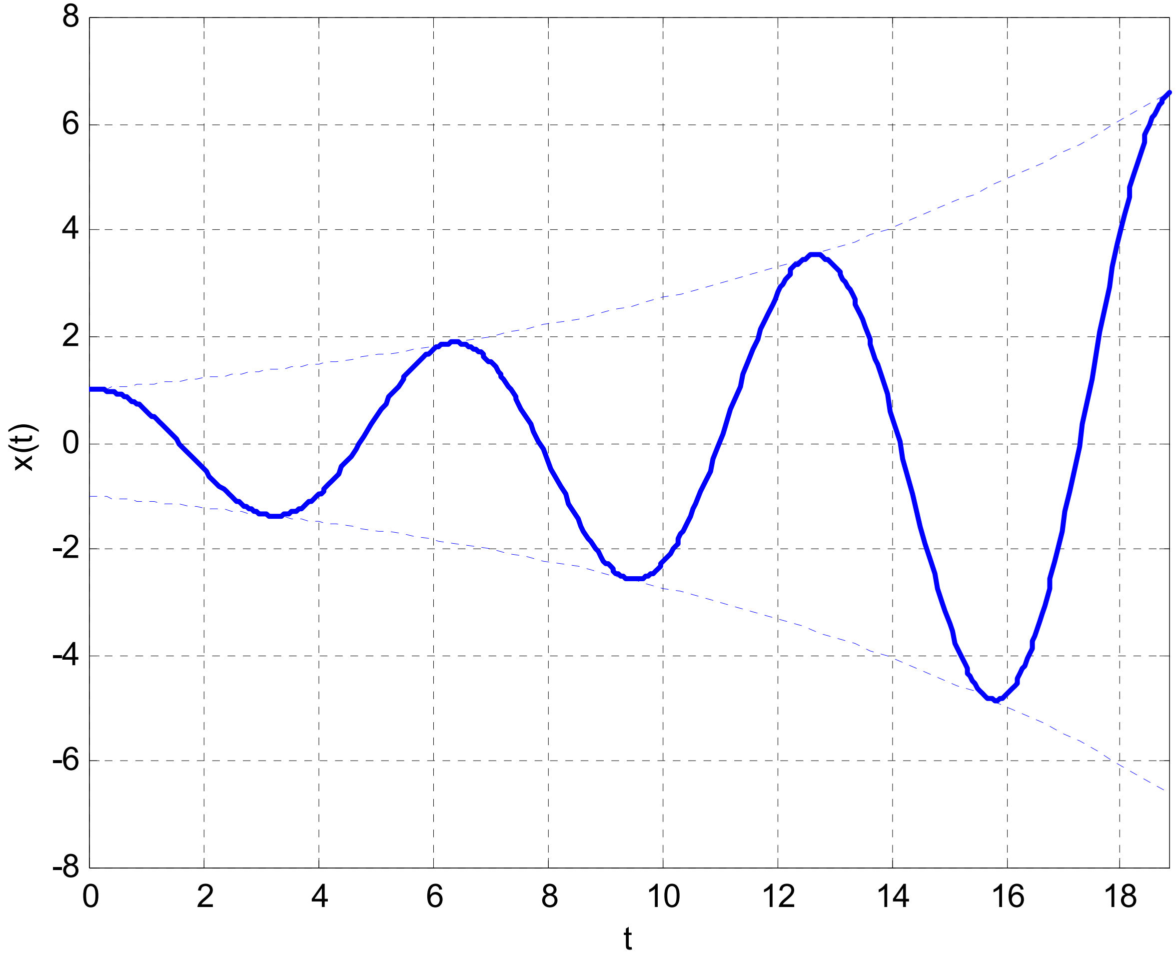
$$x(t) = X_m e^{\sigma t} \cos(\omega t + \phi)$$



$\sigma < 0$

Growing Sinusoids

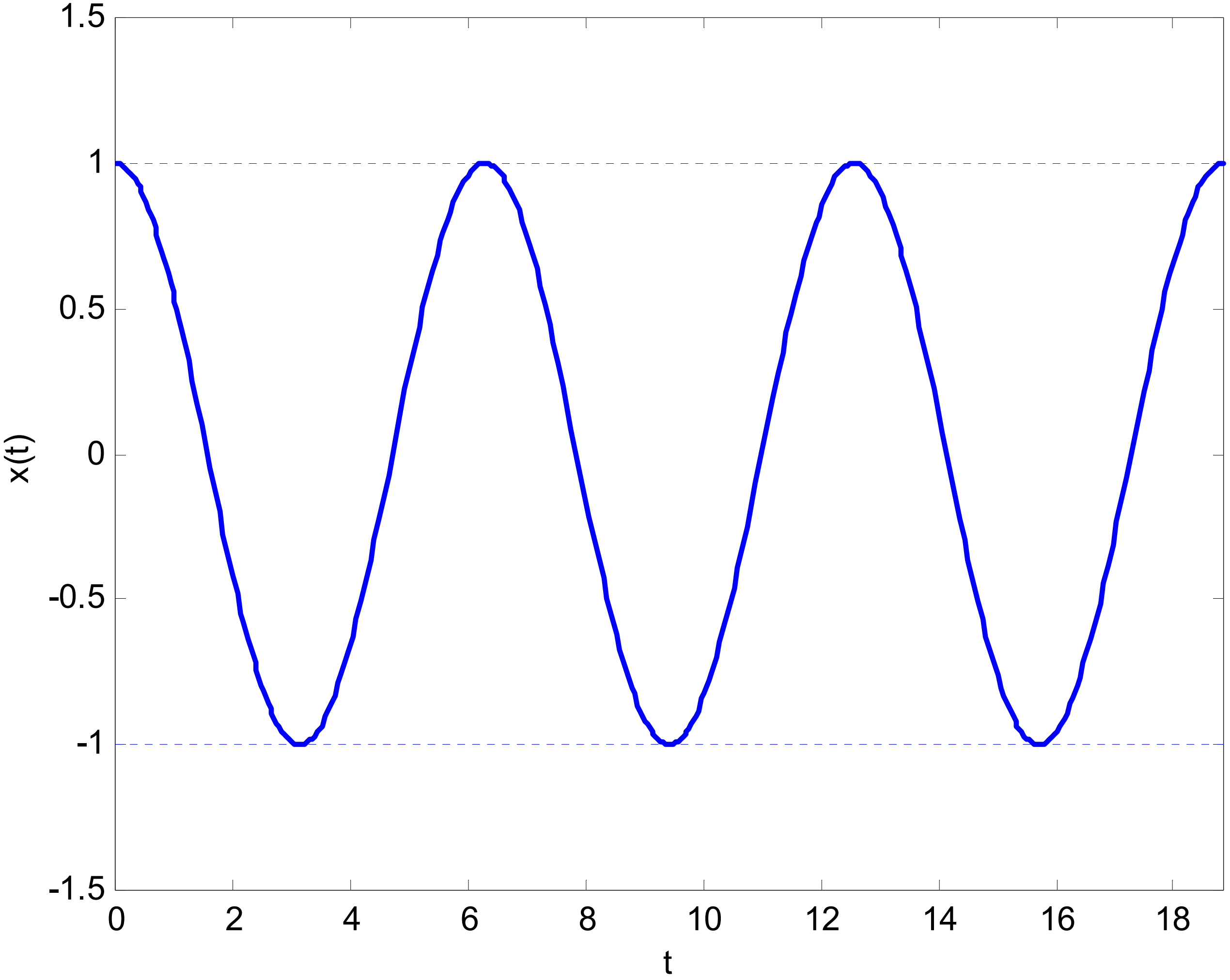
$$x(t) = X_m e^{\sigma t} \cos(\omega t + \phi)$$



$\sigma > 0$

Just Plain Sinusoids

$$x(t) = X_m e^{\sigma t} \cos(\omega t + \phi)$$



$\sigma = 0$

Generalized Phasors

Can phasors describe damped sinusoids?

Recall: For $x(t) = X_m \cos(\omega t + \phi)$

$$\mathbf{X} = X_m e^{j\phi} = X_m \angle \phi$$

$$x(t) = \mathrm{Re}[\mathbf{X} e^{j\omega t}]$$

Consider: $x(t) = X_m e^{\sigma t} \cos(\omega t + \phi)$

$$x(t) = \mathrm{Re}[X_m e^{\sigma t} e^{j(\omega t + \phi)}]$$

$$= \mathrm{Re}[X_m e^{j\phi} e^{(\sigma + j\omega)t}]$$

Thus $x(t) = \mathrm{Re}[\mathbf{X}\, e^{st}]$ where $s = \sigma + j\omega$ ← "complex" or generalized frequency

Generalized Phasors

Example:

$$x(t) = 25e^{-t}\cos(2t + 30°)$$

time domain

$$\mathbf{X}(s) = 25\angle 30^{o};$$ **s = -1 + 2j**

frequency domain

Thus

$$x(t) = \mathrm{Re}[25\angle 30° e^{st}]; \qquad s = -1 + 2j$$

Generalized Phasors

How many complex frequencies are there for:

$$x(t) = X_m e^{\sigma t}\cos(\omega t + \phi)$$

Recall: $$\cos(\omega t + \phi) = \frac{1}{2}e^{j(\omega t + \phi)} + \frac{1}{2}e^{-j(\omega t + \phi)}$$

$$x(t) = \frac{X_m}{2}e^{\sigma t}e^{j(\omega t + \phi)} + \frac{X_m}{2}e^{\sigma t}e^{-j(\omega t + \phi)}$$

$$= \frac{X_m}{2}e^{j\phi}e^{(\sigma + j\omega)t} + \frac{X_m}{2}e^{-j\phi}e^{(\sigma - j\omega)t}$$

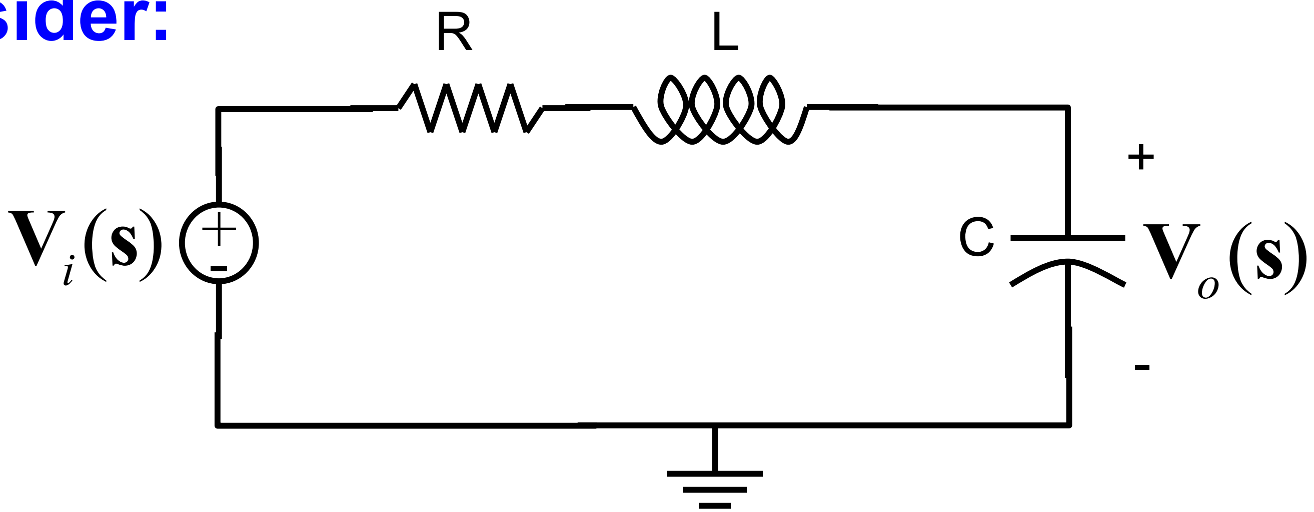
$$= \mathbf{X}e^{s_1 t} + \mathbf{X}^* e^{s_2 t}$$

Two generalized complex frequencies s_1 and s_2 such that

$$s_1 = s_2^*$$

Transfer Function/System Response

Consider:



R=6Ω

L=1H

C=0.04F

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{25}{s^2 + 6s + 25}$$

$$(s^2 + 6s + 25)V_o(s) = 25V_i(s)$$

Transfer Function/System Response

$$(\mathbf{s}^2 + 6\mathbf{s} + 25)\mathbf{V}_o(\mathbf{s}) = 25\mathbf{V}_i(\mathbf{s})$$

Question: Is there a frequency **s** such that the above system has an output when the **input is zero**?

$$\mathbf{s}^2 + 6\mathbf{s} + 25 = 0$$

homogeneous equation

$$s_1 \; -3+4j \qquad s_2 \; -3-4j$$

system poles

Thus

$$v_o(t) = \mathbf{V}_n e^{\mathbf{s}_1 t} + \mathbf{V}_n^* e^{\mathbf{s}_1 t}$$

$$= \mathbf{V}_n e^{(-3+4j)t} + \mathbf{V}_n^* e^{(-3-4j)t}$$

$$= V_n e^{j\phi} e^{(-3+4j)t} + V_n e^{-j\phi} e^{(-3-4j)t}$$

Transfer Function/System Response

$$v_o(t) = V_n e^{j\phi} e^{(-3+4j)t} + V_n e^{-j\phi} e^{(-3-4j)t}$$

$$= V_n e^{-3t}\left(e^{j(4t+\phi)} + e^{-j(4t+\phi)}\right)$$

$$= 2V_n e^{-3t}\cos(4t+\phi)$$

$$= A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t)$$

→ The poles of the system determine the **natural response**.

→ For **stable systems** the poles are on the left half of the complex plane

Transfer Function/System Response

- → The natural and forced responses of a circuit may be obtained from its network function in a straightforward manner.
- → Consider a circuit with the following I/O relationship.

$$a_n \frac{d^n x_o}{dt^n} + a_{n-1} \frac{d^{n-1} x_o}{dt^{n-1}} + \cdots + a_1 \frac{dx_o}{dt} + a_0 =$$

$$= b_m \frac{d^m x_i}{dt^n} + b_{n-1} \frac{d^{m-1} x_i}{dt^{m-1}} + \cdots + b_1 \frac{dx_i}{dt} + b_0$$

Note: RHS terms = forcing function for ODE

Transfer Function/System Response

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{X}_o(\mathbf{s})}{\mathbf{X}_i(\mathbf{s})} = \frac{b_m\mathbf{s}^m + b_{m-1}\mathbf{s}^{m-1} + \cdots + b_1\mathbf{s} + b_o}{a_n\mathbf{s}^n + a_{n-1}\mathbf{s}^{n-1} + \cdots + a_1\mathbf{s} + a_o}$$

Hence, the characteristic polynomial (of unforced system):

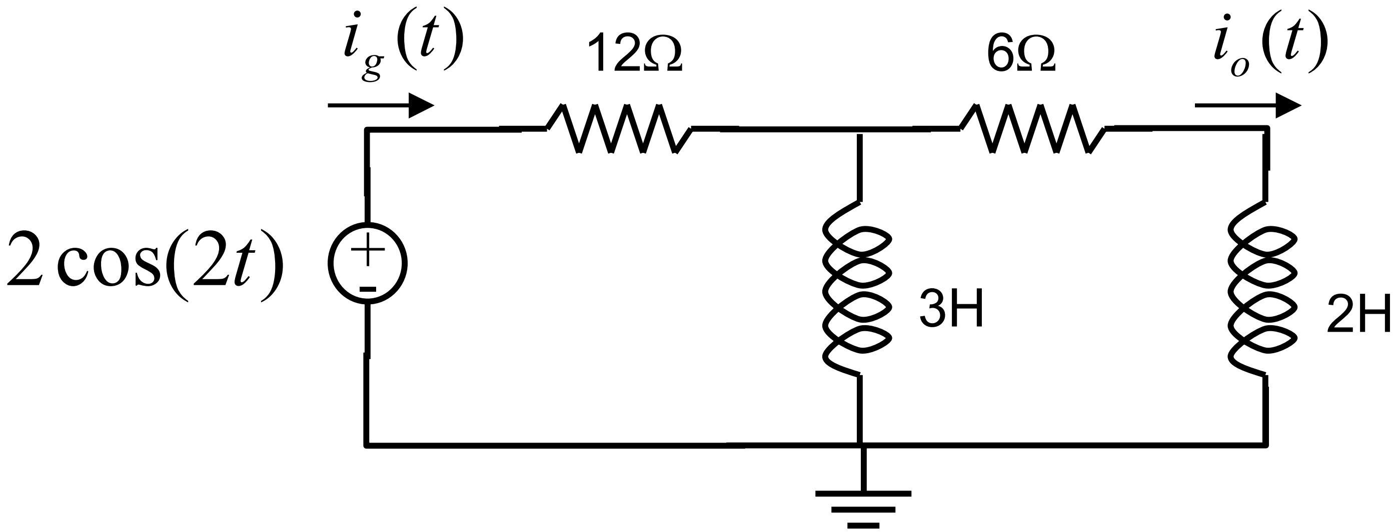
$$a_n\mathbf{s}^n + a_{n-1}\mathbf{s}^{n-1} + \cdots + a_1\mathbf{s} + a_o = 0$$

Roots of the characteristic polynomial (poles of H(s)) yield the general solution to the homogeneous equation (natural response).

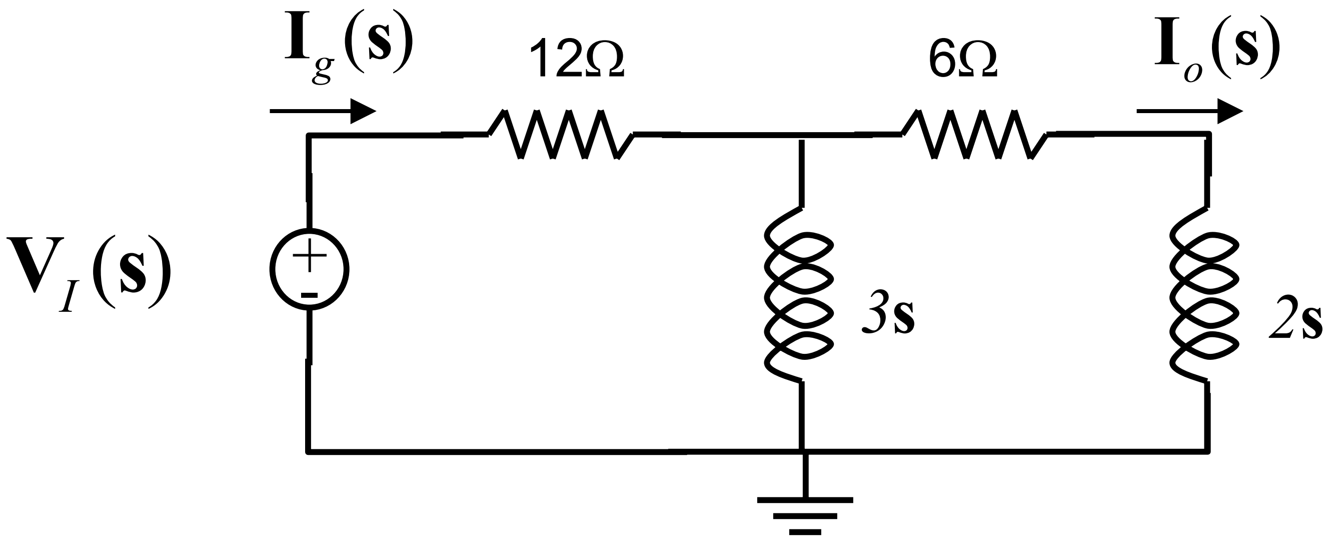
$$x_n(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}$$

Example

Assuming zero initial conditions, find $i_o(t)$.



Example



$$\mathbf{I}_g(\mathbf{s}) = \frac{\mathbf{V}_I(\mathbf{s})}{12 + 3\mathbf{s} \,||\, (6 + 2\mathbf{s})} = \frac{(6 + 5\mathbf{s})\mathbf{V}_I(\mathbf{s})}{6\mathbf{s}^2 + 78\mathbf{s} + 72}$$

$$\mathbf{I}_0(\mathbf{s}) = \frac{3\mathbf{s}}{6 + 5\mathbf{s}}\mathbf{I}_g(\mathbf{s}) = \frac{\mathbf{s}\mathbf{V}_I(\mathbf{s})}{2(\mathbf{s}+1)(\mathbf{s}+12)}$$

Voltage divider

Example

$$\mathbf{I}_0(\mathbf{s}) = \frac{3\mathbf{s}}{6+5\mathbf{s}}\mathbf{I}_g(\mathbf{s}) = \frac{\mathbf{s}\mathbf{V}_I(\mathbf{s})}{2(\mathbf{s}+1)(\mathbf{s}+12)}$$

From the poles we can write the **natural response** in the form:

$$i_n(t) = A_1 e^{-t} + A_2 e^{-12t}$$

Example

Forced response due to the input (forcing function):

$$v_i(t) = 3\cos(2t)$$

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{I}_0(\mathbf{s})}{\mathbf{V}_I(\mathbf{s})} = \frac{\mathbf{s}}{2(\mathbf{s}+1)(\mathbf{s}+12)}$$

$$\mathbf{H}(j2) = \frac{j2}{2(j2+1)(j2+12)} = 0.0368\angle 17.1$$

$$i_f(t) = 3\times 0.0368\cos(2t+17.1^\circ)$$
$$= 0.11\cos(2t+17.1^\circ)$$

Example

Total response:

$$i(t) = i_n(t) + i_f(t) =$$

$$= A_1 e^{-t} + A_2 e^{-12t} + 0.11\cos(2t + 17.1°)$$

→ Need two boundary conditions to find the two undetermined parameters.

→ For stable systems **(poles in the left half plane)**, the forced response is the steady-state response.

Natural Frequencies

1. The natural frequencies of a circuit are the poles of the network function.
2. The natural frequencies of a circuit are the same for any response in the circuit.
3. To find the natural frequencies of a circuit you may consider any response of the circuit. (Choose the simplest one!)