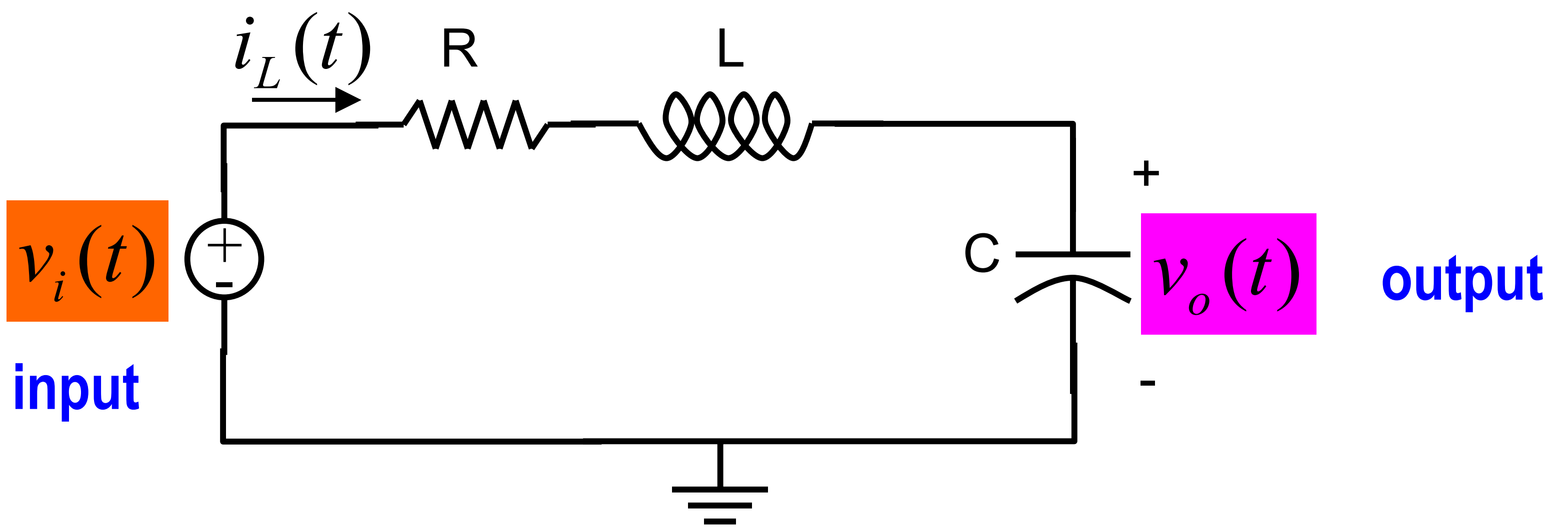
ECSE 210: Circuit Analysis

Lecture #24:

Complex Poles

Circuit Dynamics

Poles and Circuit Dynamics



$$\begin{cases} \text{KCL} \rightarrow \quad C\dfrac{dv_o}{dt} = i_L \quad \rightarrow \quad C\dfrac{d^2v_o}{dt^2} = \dfrac{di_L}{dt} \\ \text{KVL} \rightarrow \quad Ri_L + L\dfrac{di_L}{dt} + v_o = v_i \end{cases}$$

$$\Rightarrow \quad RC\frac{dv_o}{dt} + LC\frac{d^2v_o}{dt^2} + v_o = v_i$$

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R=6Ω L=1H C=0.04F

$$\Rightarrow \quad \frac{d^2 v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC} v_o = \frac{v_i}{LC}$$

$$v_i = 2\cos(3t)$$

$$\Rightarrow \quad \frac{d^2 v_o}{dt^2} + 6\frac{dv_o}{dt} + 25 v_o = 25 v_i$$

→ Second order circuit.
→ Has a **forced** and **natural** response.

The input (forcing function) is:

$$v_i = 2\cos(3t)$$

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Natural response:

R=6Ω

L=1H

C=0.04F

$$v_i = 2\cos(3t)$$

$$\Rightarrow \quad \frac{d^2 v_o}{dt^2} + 6\frac{dv_o}{dt} + 25v_o = 0$$

Characteristic polynomial: $s^2 + 6s + 25 = 0$

Roots: $$s_{1,2} = \frac{-6 \pm \sqrt{36-100}}{2} = -3 \pm 4j$$

→ Natural response of the form:

$$v_n = A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t)$$

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Forced response:

Input (forcing function): $v_i = 2\cos(3t)$

➡ Forced response of the form:

$$v_f = A_f \cos(3t) + B_f \sin(3t)$$

Total response:

$$v_o = v_n + v_f = A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t)$$
$$+ A_f \cos(3t) + B_f \sin(3t)$$

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$$v_o = v_n + v_f = A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t) + A_f\cos(3t) + B_f\sin(3t)$$

Determine the constants:

$$\frac{d^2 v_o}{dt^2} + 6\frac{dv_o}{dt} + 25v_o = 25v_i$$

$$\frac{d^2(v_f + v_n)}{dt^2} + 6\frac{d(v_f + v_n)}{dt} + 25(v_f + v_n) = 25v_i$$

$$\left(\frac{d^2 v_f}{dt^2} + 6\frac{dv_f}{dt} + 25v_f\right) + \left(\frac{d^2 v_n}{dt^2} + 6\frac{dv_n}{dt} + 25v_n\right) = 25v_i$$

0

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Substituting

$$v_f = A_f \cos(3t) + B_f \sin(3t) \quad \textbf{and} \quad v_i = 2\cos(3t)$$

into

$$\frac{d^2 v_f}{dt^2} + 6\frac{dv_f}{dt} + 25 v_f = 25 v_i$$

gives:

$$\left(-9A_f \cos(3t) - 9B_f \sin(3t)\right) + 6\left(-3A_f \sin(3t) + 3B_f \cos(3t)\right)$$

$$+25\left(A_f \cos(3t) + B_f \sin(3t)\right) = 50\cos(3t)$$

$$\left.\begin{aligned} 16A_f + 18B_f &= 50 \\ -18A_f + 16B_f &= 0 \end{aligned}\right\} \qquad \begin{aligned} A_f &= 1.38 \\ B_f &= 1.55 \end{aligned}$$

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$$\left.\begin{aligned} A_f &= 1.38 \\ B_f &= 1.55 \end{aligned}\right\} \quad v_f = 1.38\cos(3t) + 1.55\sin(3t)$$

Equivalent more compact form:

$$v_f = V_f\cos(3t+\theta_f) = V_f\cos(\theta_f)\cos(3t) - V_f\sin(\theta_f)\sin(3t)$$

$$V_f = \sqrt{1.38 + 1.55} = 2.07$$

$$\left.\begin{aligned} 2.07\cos(\theta_f) &= 1.38 \\ 2.07\sin(\theta_f) &= -1.55 \end{aligned}\right\} \quad \theta_f = \arctan\left(-\frac{1.55}{1.38}\right) = -48.3^o$$

$$v_f = 2.07\cos(3t - 48.3^o)$$

→ Forced response

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$$v_o = v_n + v_f = A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t)$$
$$+1.38\cos(3t) + 1.55\sin(3t)$$

Determine the constants **(zero initial conditions):**

$$v_o(0) = A_n + 1.38 = 0 \quad \rightarrow \quad A_n = -1.38$$

$$\frac{dv_o(0)}{dt} = -3A_n + 4B_n + 3\times 1.55 = -3A_n + 4B_n + 4.65$$

<u>KCL</u> $$C\frac{dv_o(0)}{dt} = i_L(0) = 0$$

$$-3(-1.38) + 4B_n + 4.65 = 0$$

$$\Rightarrow \quad B_n = -2.2$$

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Natural response

$$v_o = v_n + v_f = \overbrace{-1.38e^{-3t}\cos(4t) - 2.2e^{-3t}\sin(4t)}^{\text{Natural response}}$$

$$\underbrace{+1.38\cos(3t) + 1.55\sin(3t)}_{\text{Forced response}}$$

Forced response

At $t=\infty$, the **natural** response dies out. Therefore :
- The **forced** response is the **steady-state** response.

- The **transient** response is due to both the forced and natural response.

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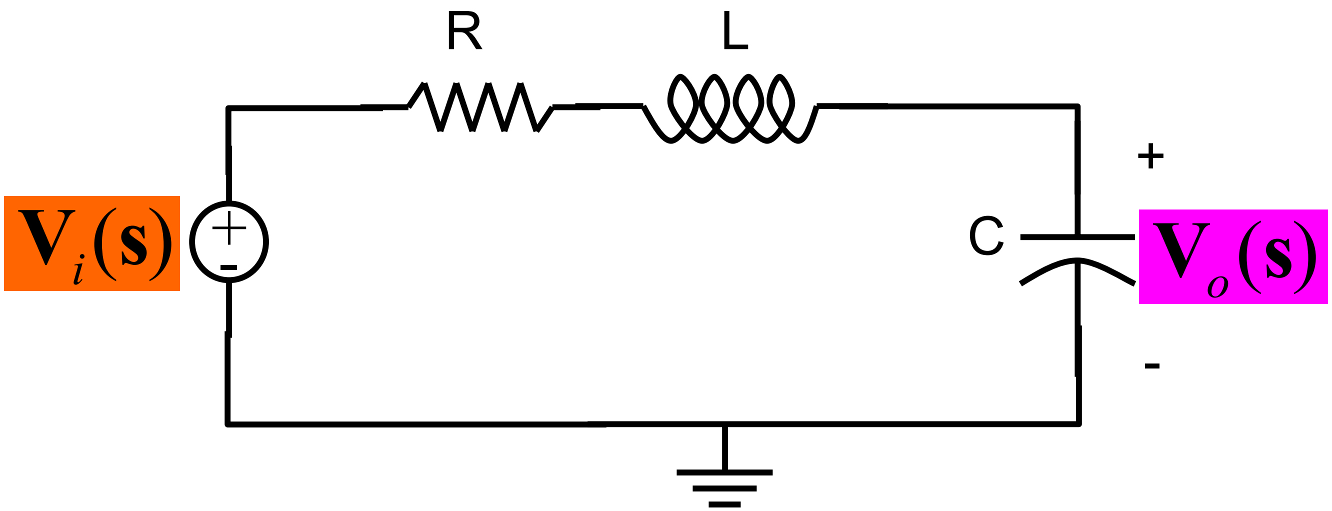

$$\mathbf{V}_o(\mathbf{s}) = \frac{{}^{1}\!/_{\mathbf{s}C}}{R + \mathbf{s}L + {}^{1}\!/_{\mathbf{s}C}} \mathbf{V}_i(\mathbf{s}) = \frac{1}{LC\mathbf{s}^2 + RC\mathbf{s} + 1} \mathbf{V}_i(s)$$

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{{}^{1}\!/_{LC}}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}}$$

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$$\left(\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}\right)\mathbf{V}_o(\mathbf{s}) = \frac{1}{LC}\mathbf{V}_i(\mathbf{s})$$

$$\Downarrow$$

$$\frac{d^2 v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC}v_o = \frac{v_i}{LC}$$

Same as the differential equation we derived in the time domain.

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What is the **steady-state** response due to $v_i = 2\cos(3t)$?

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{1/LC}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}}$$

R=6Ω

L=1H

C=0.04F

$$\mathbf{H}(\mathbf{s}) = \frac{25}{\mathbf{s}^2 + 6\mathbf{s} + 25}$$

$$\mathbf{H}(j3) = \frac{25}{(j3)^2 + 6(j3) + 25} = 1.03\angle -48^o$$

Same result obtained using ODE approach.

Steady state response:

$$v_o = 2\,|\,\mathbf{H}(j3)\,|\cos(3t + \angle\mathbf{H}(j3)) = 2.06\cos(3t - 48^o)$$

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$$v_o = v_n + v_f = A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t)$$

$$+1.38\cos(3t) + 1.55\sin(3t)$$

or

$$v_o = v_n + v_f = A_n e^{-3t}\cos(4t) + B_n e^{-3t}\sin(4t)$$

$$+2.07\cos(3t - 48.3^o)$$

Next step:

Determine the constants **(zero initial conditions).**