ECSE 210: Circuit Analysis

Lecture #23:

Bode Plots

Complex Poles

Complex Poles and Zeros

In general, network functions can be expressed as the ratio of two polynomials in **s**:

$$\mathbf{H}(\mathbf{s}) = \frac{N(\mathbf{s})}{D(\mathbf{s})} = \frac{a_m \mathbf{s}^m + a_{m-1}\mathbf{s}^{m-1} + \cdots + a_1 \mathbf{s} + a_o}{b_n \mathbf{s}^n + b_{n-1}\mathbf{s}^{n-1} + \cdots + b_1 \mathbf{s} + b_o}$$

$$= \frac{K_o(\mathbf{s} - z_1)(\mathbf{s} - z_2)\cdots(\mathbf{s} - z_m)}{(\mathbf{s} - p_1)(\mathbf{s} - p_2)\cdots(\mathbf{s} - p_n)}$$

→ The roots of the numerator and denominator can be complex, resulting in **complex poles and zeros**.

Complex Poles and Zeros

$$\mathbf{H}(s) = \frac{1}{s^2 + 2\zeta\omega_o s + \omega_o^2}$$

The roots of the denominator are:

$$p_{1,2} = -\zeta\omega_o \pm \omega_o\sqrt{\zeta^2 - 1}$$

$\zeta > 1$ ⇒ Two real poles

$\zeta = 1$ ⇒ Two real and equal poles (a pole of multiplicity 2)

$\zeta < 1$ ⇒ Two complex poles

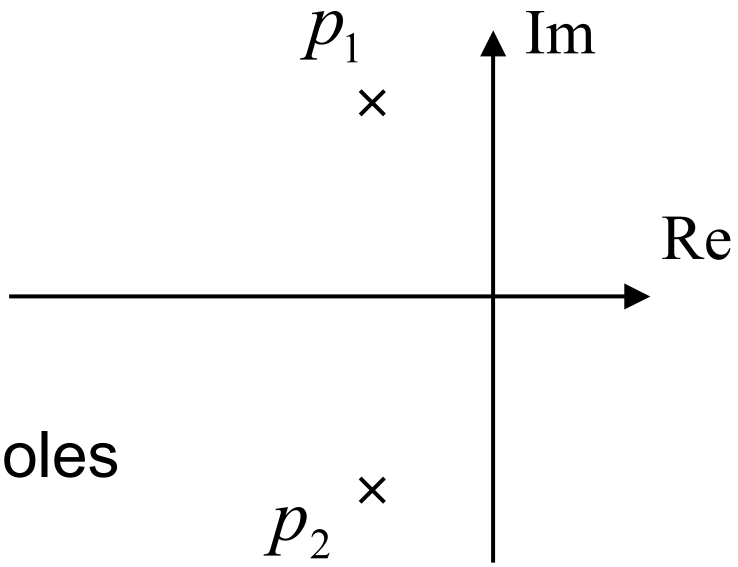
Complex Poles and Zeros

Example

$$\mathbf{H(s)} = \frac{1}{\mathbf{s}^2 + 10\mathbf{s} + 100} \qquad \begin{cases} \omega_o = 10 \\ \zeta = 0.5 \end{cases}$$

$$p_{1,2} = -\zeta\omega_o \pm \omega_o\sqrt{\zeta^2 - 1} = -5 \pm j10\sqrt{1 - 0.25}$$

$$p_{1,2} = -5 \pm j8.6$$



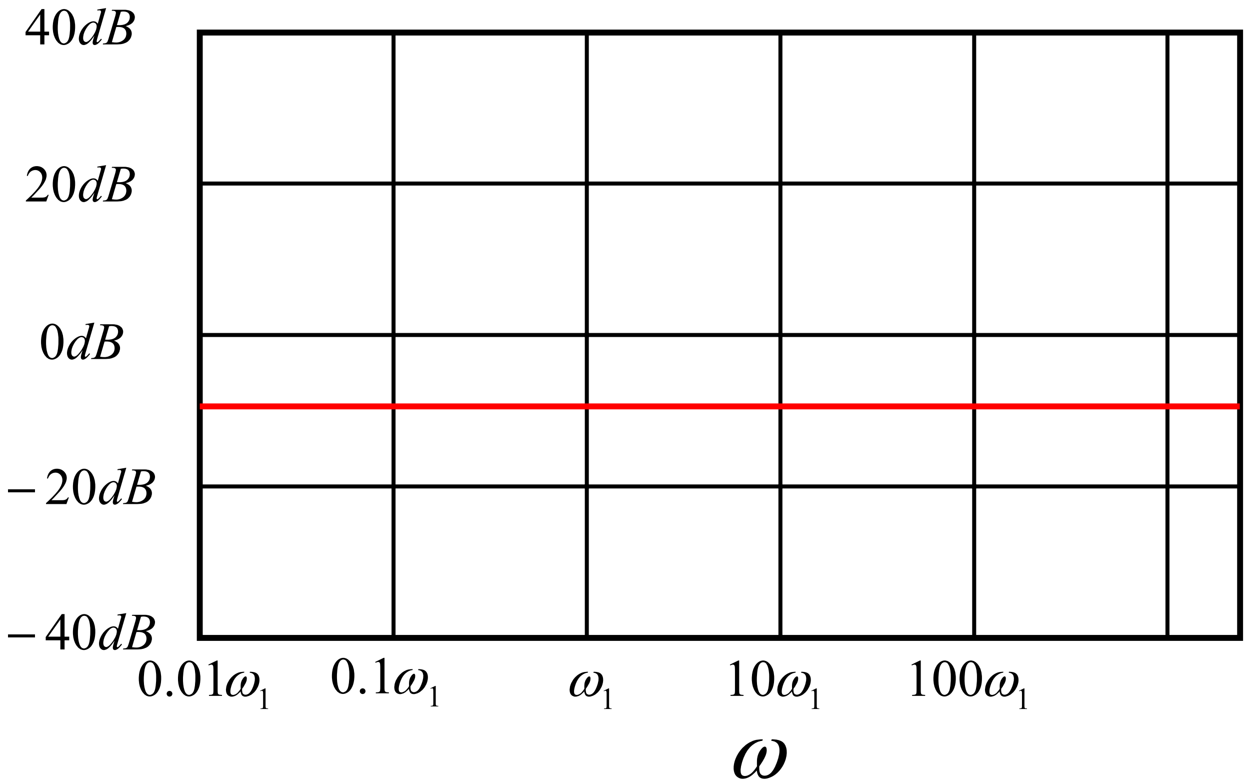
$$\mathbf{H(s)} = \frac{1}{(\mathbf{s} + 5 + j8.6)(\mathbf{s} + 5 - j8.6)}$$

→ Treat the two complex conjugate poles simultaneously as a pair.

Complex Poles and Zeros

Low frequency asymptote:

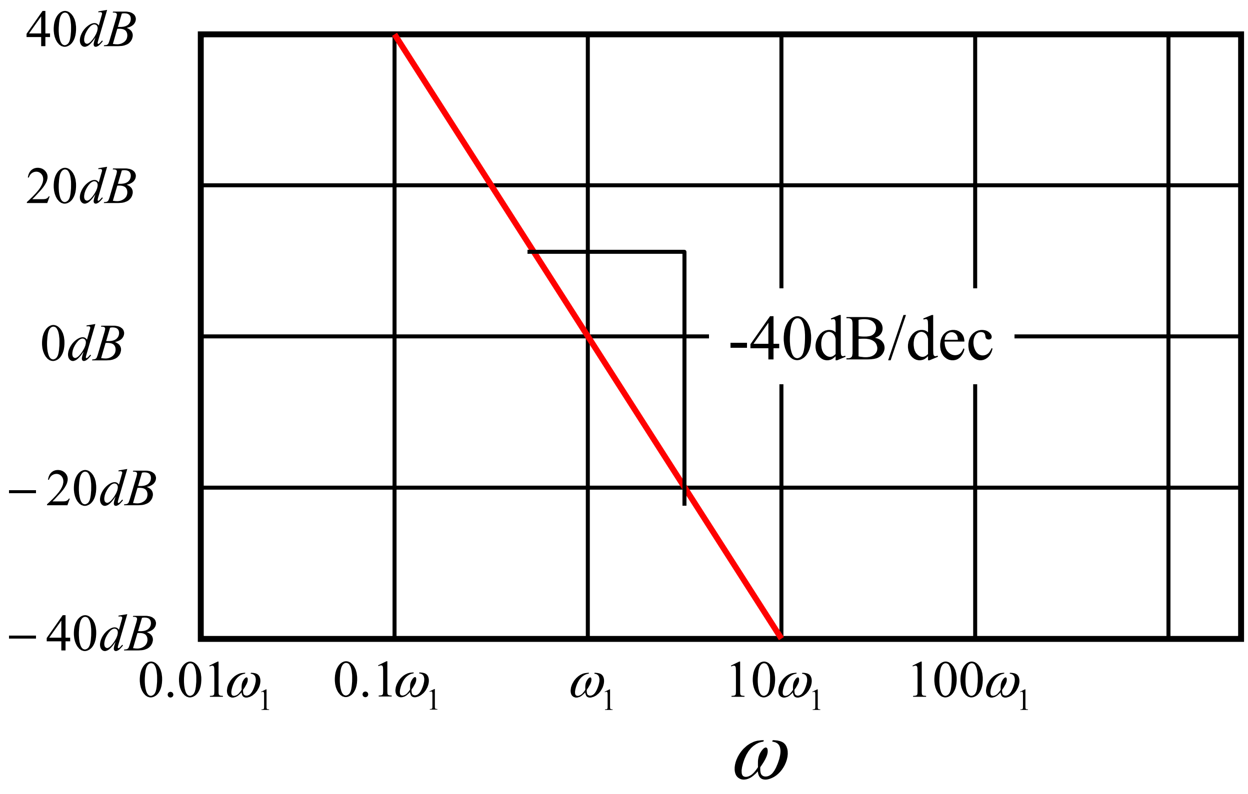
$$\omega \ll \omega_o \Rightarrow \left|\mathbf{H}(j\omega)\right|_{dB} \approx 20\log\left(\frac{1}{\omega_o^2}\right)$$



Complex Poles and Zeros

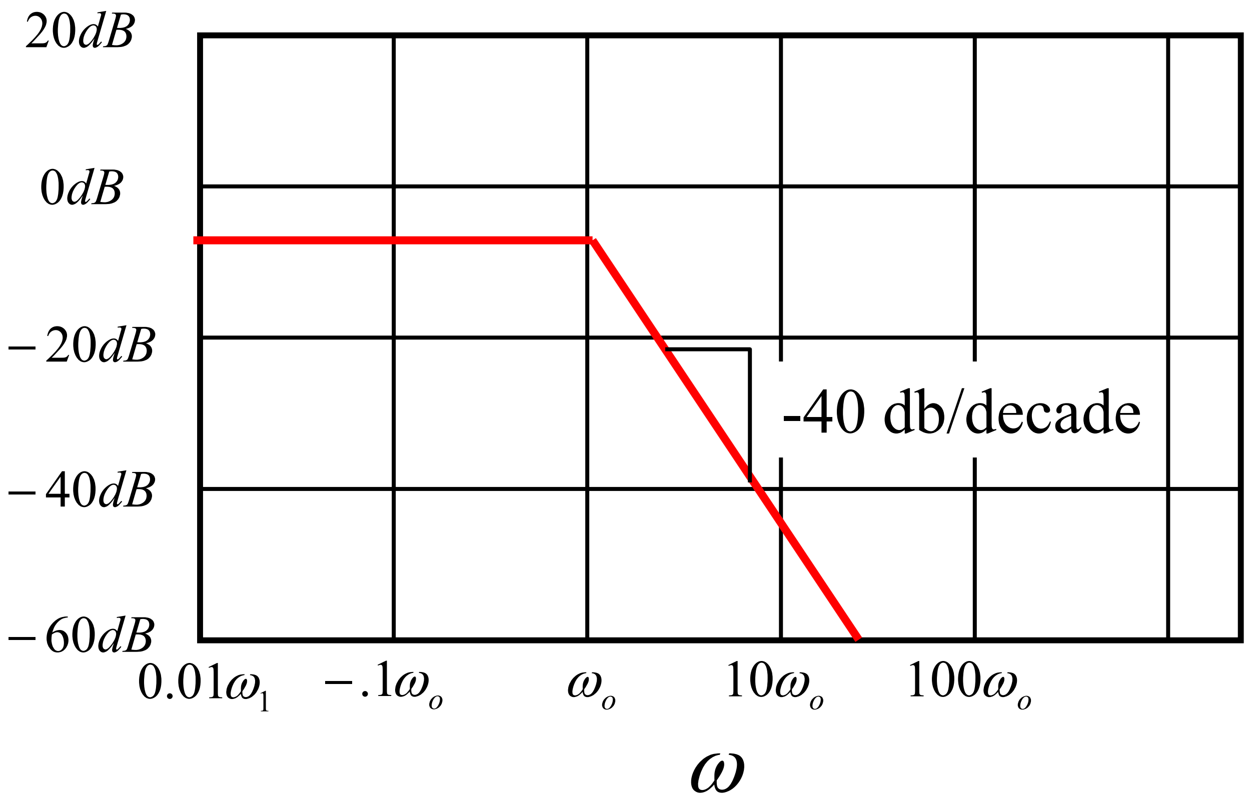
High frequency asymptote:

$$\omega >> \omega_o \Rightarrow \left|\mathbf{H}(j\omega)\right|_{dB} \approx 20\log\left(\frac{1}{\omega^2}\right) = -40\log(\omega)$$



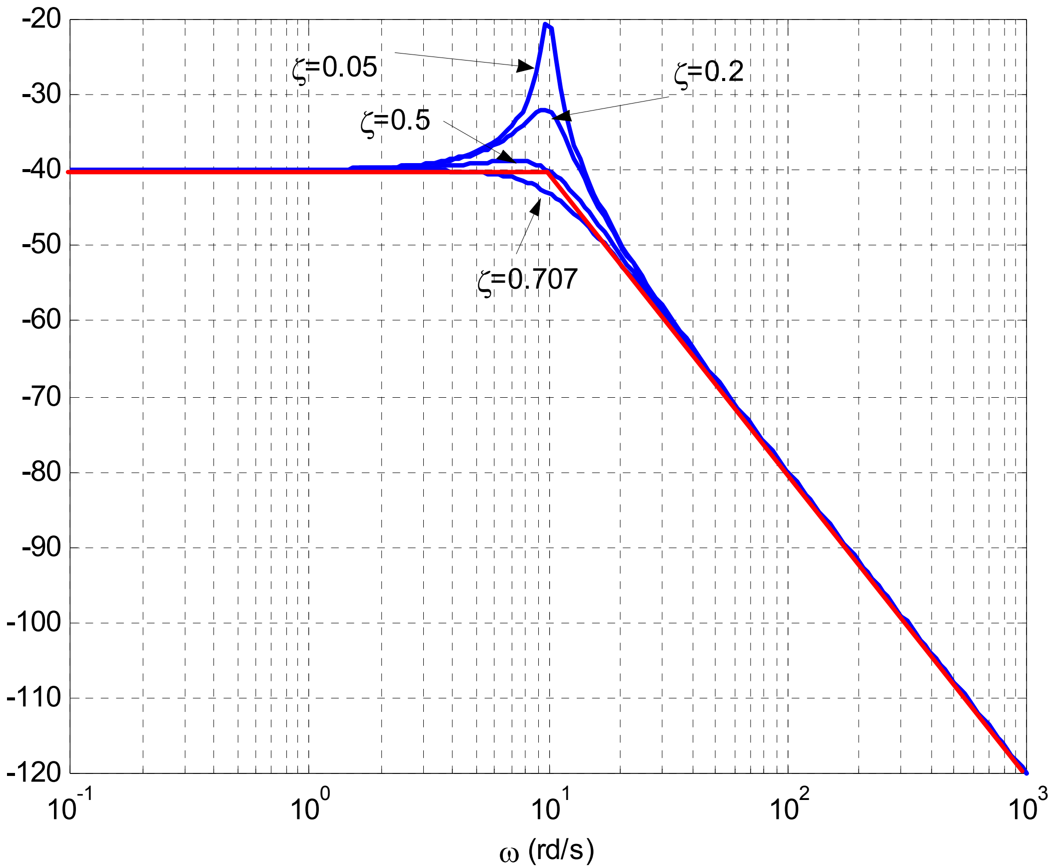
Complex Poles and Zeros

Note that the two asymptotes intersect at $\omega = \omega_o$



Example

$$\mathbf{H(s)} = \frac{1}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2} \qquad \omega_o = 10\,rad./\sec.$$



Complex Poles

Maximum value of $|\mathbf{H}(j\omega)|$

$$\mathbf{H}(\mathbf{s}) = \frac{1}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$

$$p_{1,2} = -\zeta\omega_o \pm \omega_o\sqrt{\zeta^2 - 1}$$

$$\zeta < 1 \quad \Rightarrow \quad p_{1,2} = -\zeta\omega_o \pm j\omega_o\sqrt{1-\zeta^2}$$

$$\mathbf{H}(j\omega) = \frac{1}{\left(j\omega + \zeta\omega_o + j\omega_o\sqrt{1-\zeta^2}\right)\left(j\omega + \zeta\omega_o - j\omega_o\sqrt{1-\zeta^2}\right)}$$

Complex Poles

Maximum value of $|\mathbf{H}(j\omega)|$

$$\mathbf{H}(\mathbf{s}) = \frac{1}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2} \qquad \mathbf{H}(j\omega) = \frac{1}{\omega_o^2 - \omega^2 + j2\zeta\omega_o\omega}$$

$$\left|\mathbf{H}(j\omega)\right| = \frac{1}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + 4\zeta^2\omega_o^2\omega^2}}$$

Take derivative and set it to zero to find ω at maximum value.

$$\frac{d}{d\omega}\left|\mathbf{H}(j\omega)\right| = 0 \quad \Leftrightarrow \quad \omega = \pm\omega_o\sqrt{1 - 2\zeta^2}$$

Complex Poles

$$\frac{d}{d\omega}\left|\mathbf{H}(j\omega)\right| = 0 \quad \Longleftrightarrow \quad \omega = \pm\omega_o\sqrt{1-2\zeta^2}$$

A *real* positive value solution for ω exists at the peak only if

$$2\zeta^2 < 1 \;\Rightarrow\; \zeta < \frac{1}{\sqrt{2}} \;\Rightarrow\; \zeta < 0.707$$

$$\omega_r = \omega_o\sqrt{1-2\zeta^2}$$

Called the resonant frequency, it is the frequency at which the response is a maximum.

$$\zeta > \frac{1}{\sqrt{2}} \;\Rightarrow$$

There is no resonant frequency and the response decays monotonically.

Complex Poles

$$\mathbf{H}(\mathbf{s}) = \frac{1}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$

$$\omega_r = \omega_o\sqrt{1-2\zeta^2}$$ **Resonant frequency**

$$\left|\mathbf{H}(j\omega_r)\right| = \frac{1}{2\zeta\omega_o^2\sqrt{1-\zeta^2}}$$

$$\left|\mathbf{H}(j0)\right| = \frac{1}{\omega_o^2}$$

$$\frac{\left|\mathbf{H}(j\omega_r)\right|}{\left|\mathbf{H}(j0)\right|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Example 1

Draw the amplitude Bode plot for:

$$\mathbf{H}(\mathbf{s}) = \frac{100}{\mathbf{s}^2 + 50\mathbf{s} + 10000}$$

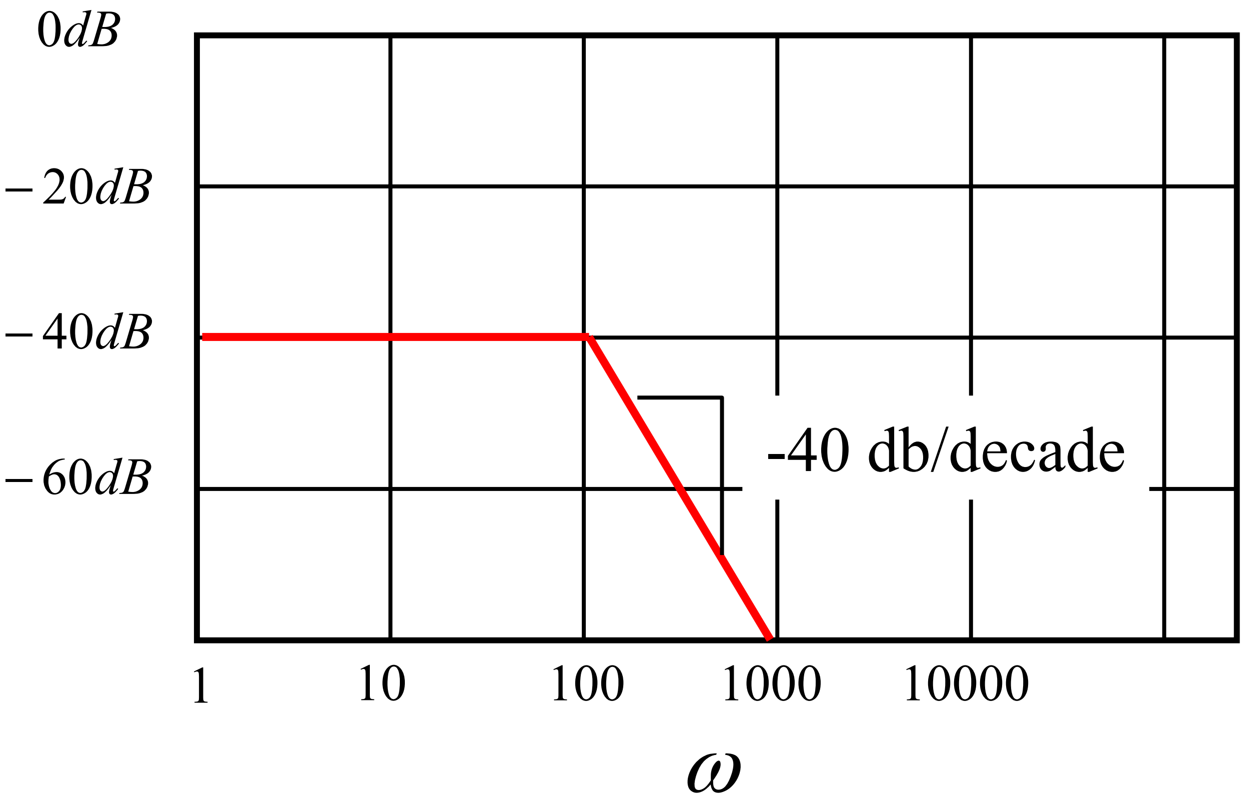
$$\mathbf{H}(\mathbf{s}) = \frac{k}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$ → Standard form

$$\omega_o = 100 rad./\sec.$$

$$2\zeta\omega_o = 50 \qquad \zeta = \frac{50}{2\times 100} = 0.25$$

Example 1: Asymptotes

$$\omega << 100 \quad \Rightarrow \quad 20\log\left(\left|\mathbf{H}(j\omega)\right|\right) \cong 20\log\left(\frac{100}{10000}\right) = -40$$



Example 1: Asymptotes

Since $\zeta = 0.25 < 0.707$ → There is resonance.

The frequency at which resonance occurs (i.e., where $|\mathbf{H}(j\omega)|$ is a maximum) is given by:

$$\omega_r = \omega_o \sqrt{1 - 2\zeta^2} = 100\sqrt{1 - 0.125} = 93.5\, rad./sec.$$

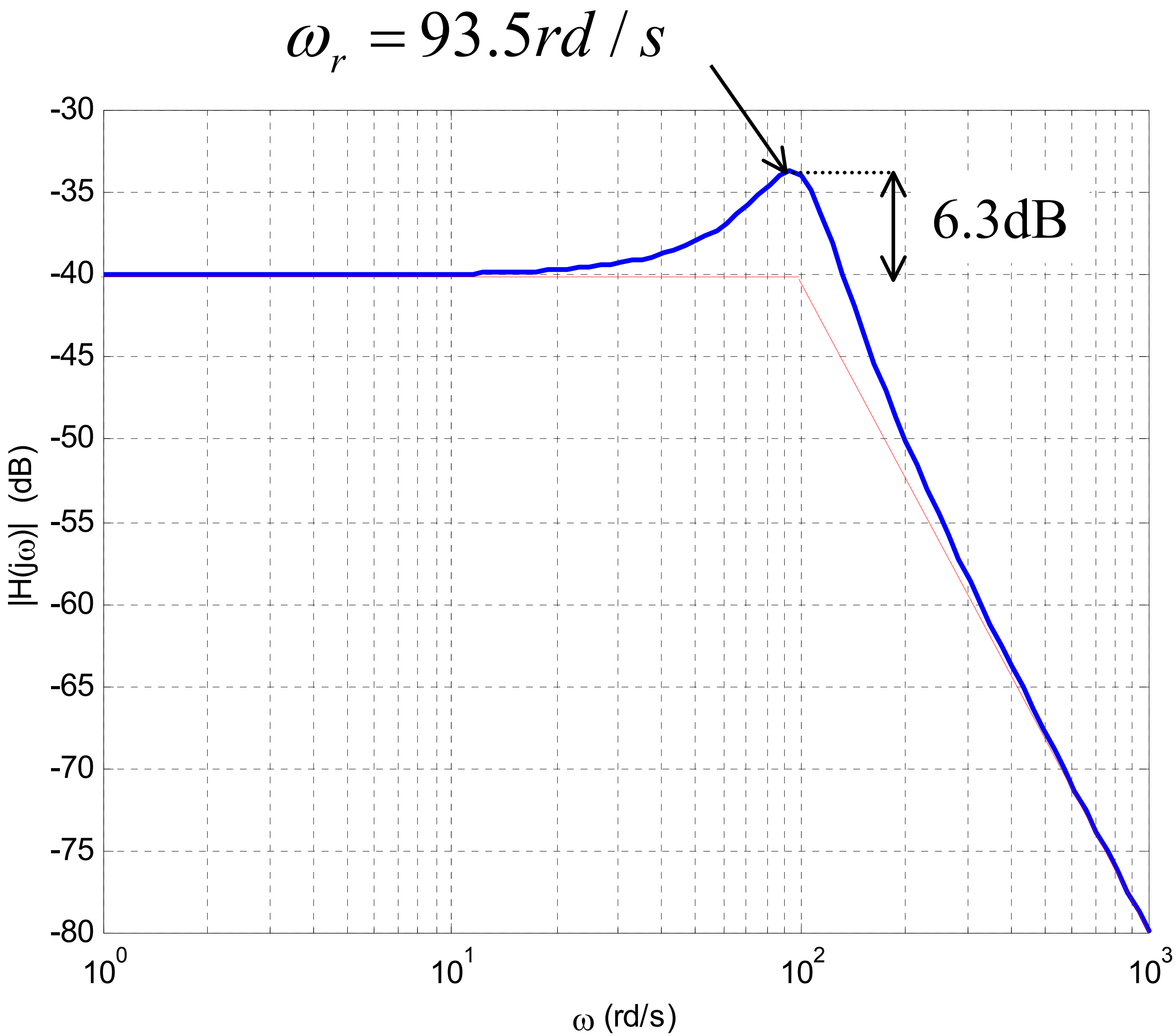
Note: ω_o = 100 rad./sec.

At $\omega = \omega_r = 93.5$ *rad/sec.* we have:

$$\frac{|\mathbf{H}(j\omega_r)|}{|\mathbf{H}(j0)|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{0.5\sqrt{1-0.0625}} = 2.066$$

$$20\log(2.066) = 6.3dB$$

Example 1: Corrected Response



Example 2

Draw the amplitude Bode plot of the following transfer function:

$$\mathbf{H(s)} = \frac{10000}{\mathbf{s}^2 + 40\mathbf{s} + 40000}$$

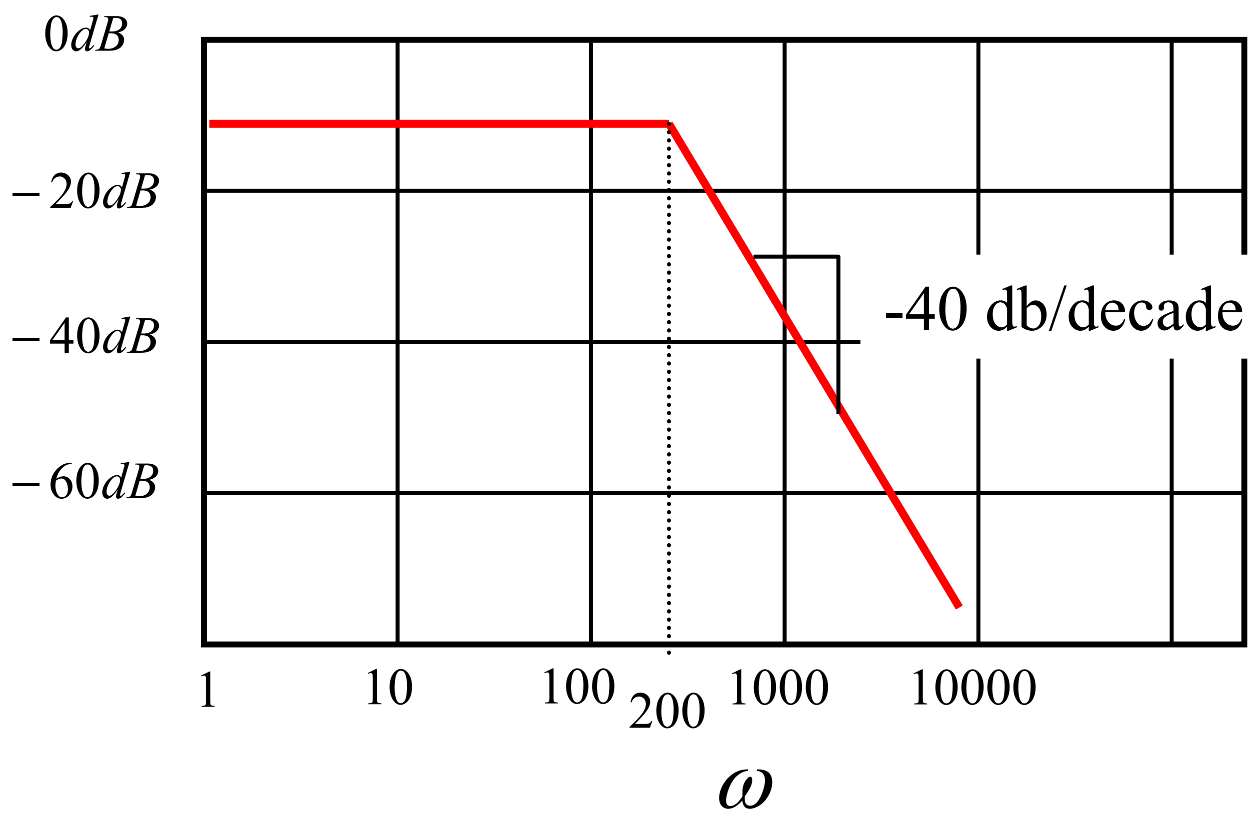
$$\mathbf{H(s)} = \frac{k}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$ → Standard form

$$\omega_o = 200 rad./sec.$$

$$2\zeta\omega_o = 40 \qquad \zeta = \frac{40}{2 \times 200} = 0.1$$

Example 2: Asymptotes

$$\omega << 200 \Rightarrow 20\log\left(\left|\mathbf{H}(j\omega)\right|\right) \cong 20\log\left(\frac{10000}{40000}\right) = -12dB$$



Example 2: Asymptotes

Since $\zeta = 0.1 < 0.707$ → There is resonance.

The frequency at which resonance occurs (i.e., where $|\mathbf{H}(j\omega)|$ is a maximum is given by:

$$\omega_r = \omega_o\sqrt{1-2\zeta^2} = 200\sqrt{1-0.02} = 198 rd/s$$

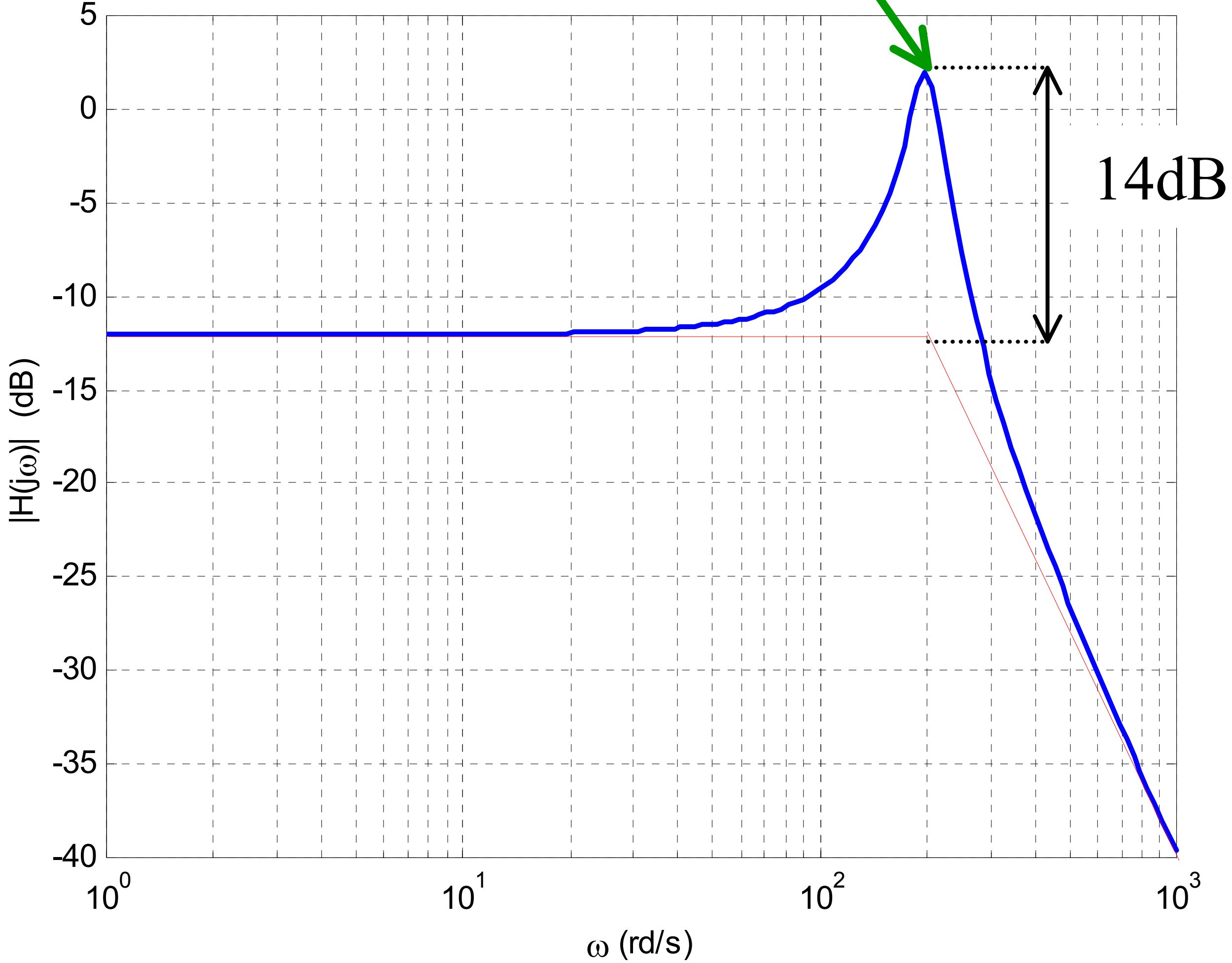
At $\omega = \omega_r$=198 *rad./sec.* we have:

$$\frac{|\mathbf{H}(j\omega_r)|}{|\mathbf{H}(j0)|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{0.2\sqrt{1-0.01}} = 5.025$$

$$20\log(5.025) = 14dB$$

Example 2: Corrected Response

$\omega_r = 198 \quad rad./\sec.$



Example 3

Draw the amplitude Bode plot of the following transfer function:

$$\mathbf{H(s)} = \frac{10000}{\mathbf{s}^2 + 100\mathbf{s} + 10000}$$

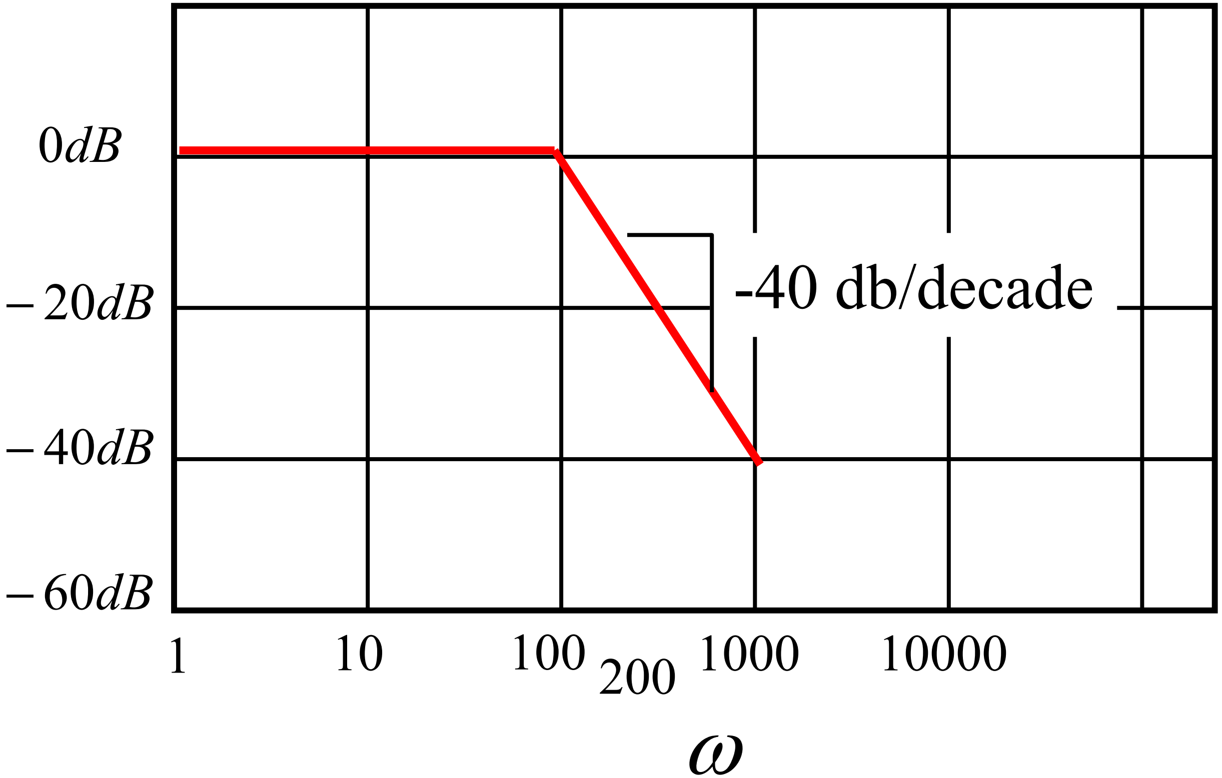
$$\mathbf{H(s)} = \frac{1}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$ → Standard form

$$\omega_o = 100 rad./sec.$$

$$2\zeta\omega_o = 100 \qquad \zeta = \frac{100}{2 \times 100} = 0.5$$

Example 3: Asymptotes

$$\omega << 100 \quad \Rightarrow \quad 20\log\left(\left|\mathbf{H}(j\omega)\right|\right) \cong 20\log\left(\frac{10000}{10000}\right) = 0dB$$



Example 3: Asymptotes

Since $\zeta = 0.5 < 0.707$ → There is resonance.

The frequency at which resonance occurs (i.e., where $|\mathbf{H}(j\omega)|$ is maximum is given by:

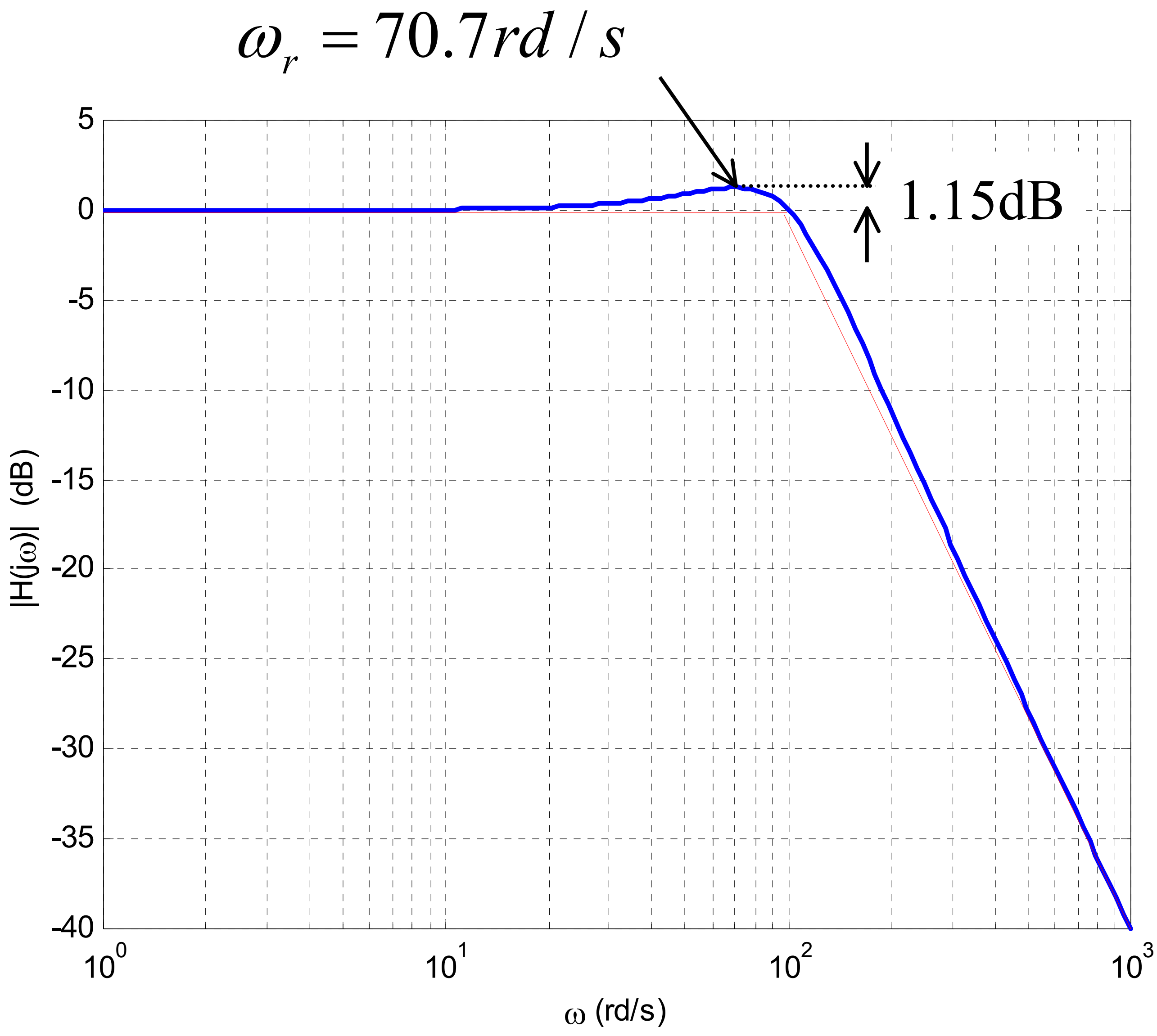
$$\omega_r = \omega_o\sqrt{1-2\zeta^2} = 100\sqrt{1-0.5} = 70.7\,rad./sec.$$

At $\omega = \omega_r = 86$ *rad./sec.* we have:

$$\frac{|\mathbf{H}(j\omega_r)|}{|\mathbf{H}(j0)|} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{1\sqrt{1-0.25}} = 1.15$$

$$20\log(1.15) = 1.25dB$$

Example 3: Corrected Response



Example 4

Draw the amplitude Bode plot of the following transfer function:

$$\mathbf{H}(\mathbf{s}) = \frac{10000}{\mathbf{s}^2 + 160\mathbf{s} + 10000}$$

$$\mathbf{H}(\mathbf{s}) = \frac{1}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$ → Standard form

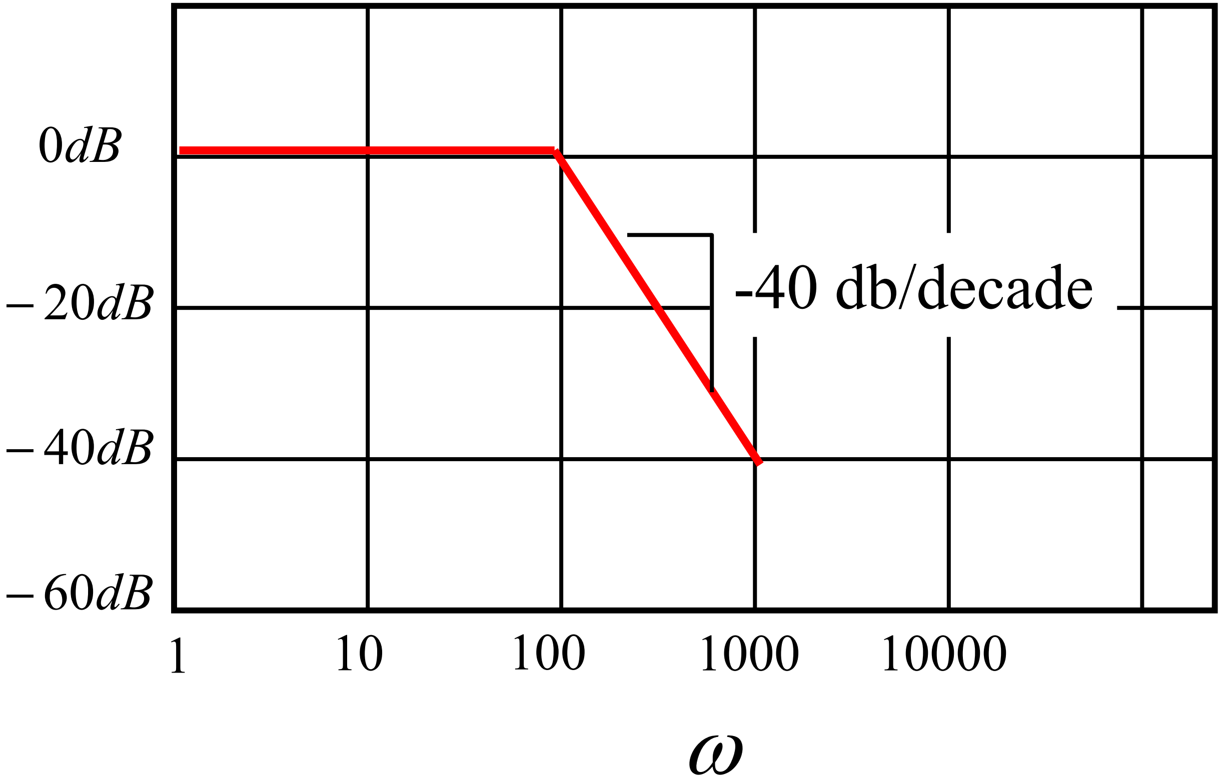
$$\omega_o = 100 rad./sec.$$

$$2\zeta\omega_o = 160 \qquad \zeta = \frac{160}{2\times 100} = 0.8$$

There is no resonance.

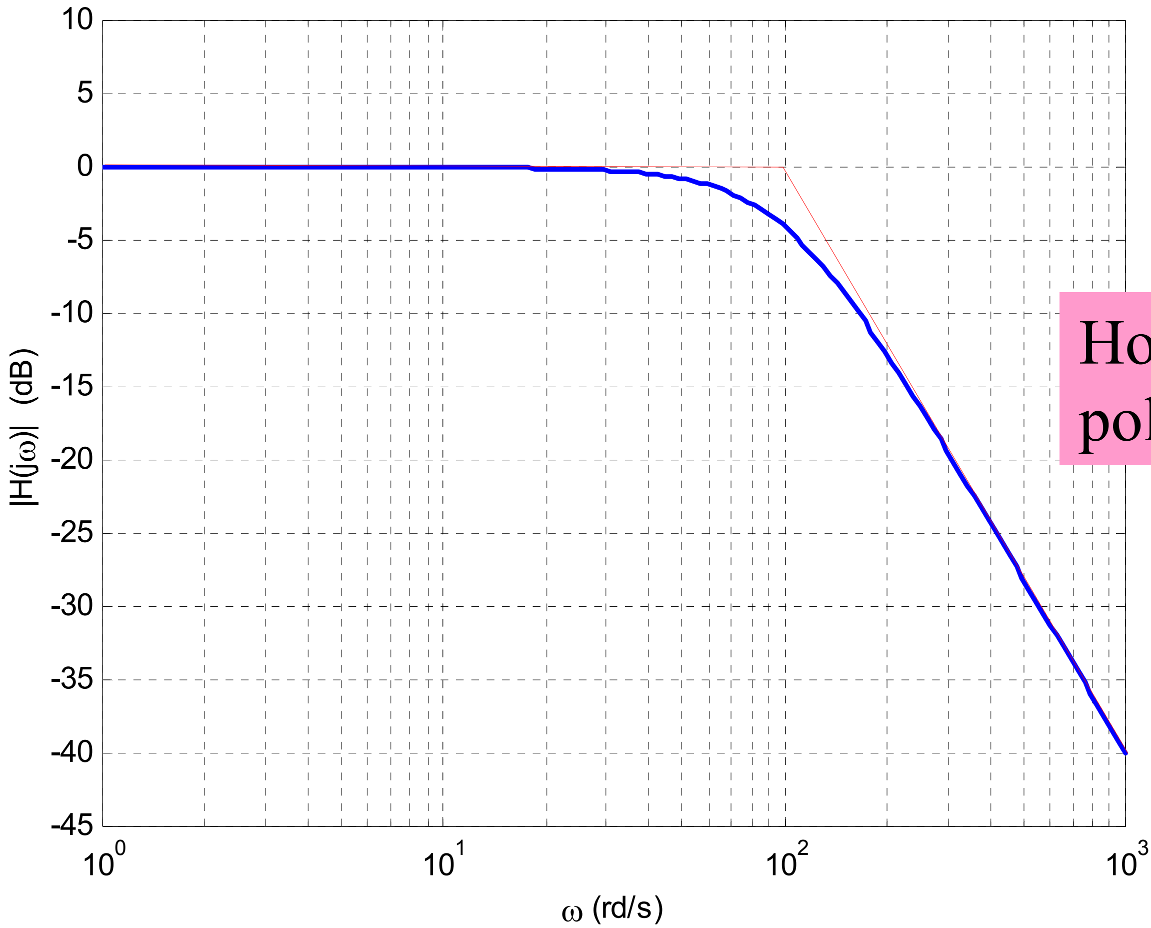
Example 4: Asymptotes

$$\omega << 100 \quad \Rightarrow \quad 20\log\left(\left|\mathbf{H}(j\omega)\right|\right) \cong 20\log\left(\frac{10000}{10000}\right) = 0dB$$



Example 4: Corrected Response

Since $\zeta = 0.8 > 0.707$ → There is no resonance.



How are the poles different?