ECSE 210: Circuit Analysis

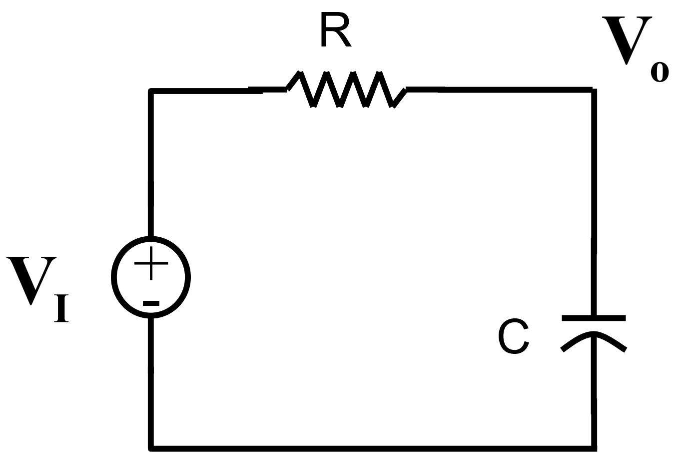
Lecture #21:

Frequency Response

Decibel Scale

Example

What is V_o as a function of V_I?



$$\mathbf{Z}_C = \frac{1}{\mathbf{s}C}$$

$$\mathbf{s} = j\omega$$

$$\mathbf{Z}_R = R$$

→ Voltage divider:

$$\mathbf{V_o} = \frac{\frac{1}{\mathbf{s}C}}{R + \frac{1}{\mathbf{s}C}}\mathbf{V_I} = \frac{1}{1 + \mathbf{s}RC}\mathbf{V_I} = \frac{1}{1 + j\omega RC}\mathbf{V_I}$$

Frequency Characteristics

$$\mathbf{H(s)} = \frac{\mathbf{V_o}}{\mathbf{V_I}} = \frac{1}{1+\mathbf{s}RC} = \frac{1/RC}{\mathbf{s}+1/RC}$$

$R = 1k\Omega$ $\quad \omega = 2\pi f$

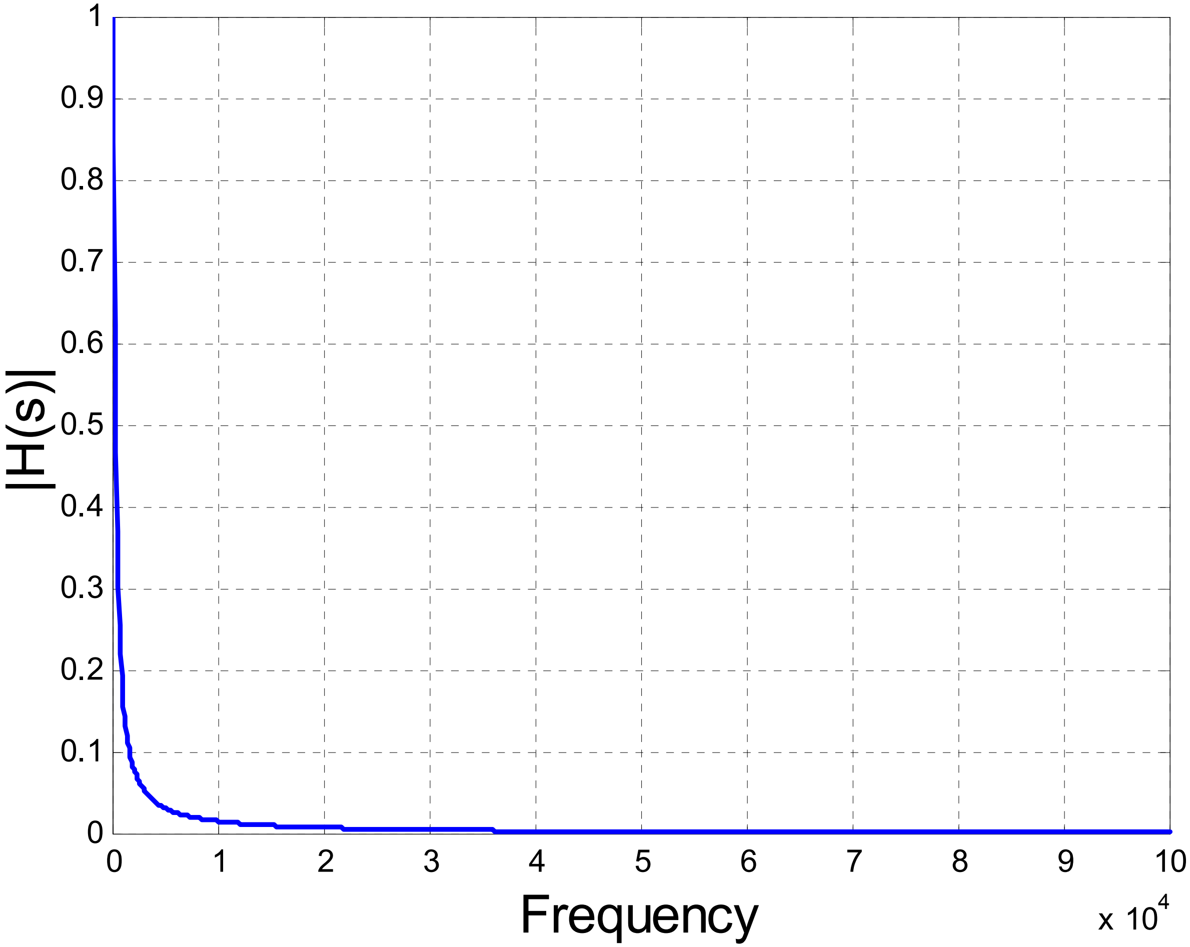
$C = 1\mu F$

$$\mathbf{H(s)} = \frac{\mathbf{V_o}}{\mathbf{V_I}} = \frac{1}{1+\mathbf{s}RC} = \frac{1000}{\mathbf{s}+1000}$$

$$\mathbf{H}(j\omega) = \frac{1000}{j\omega+1000}$$

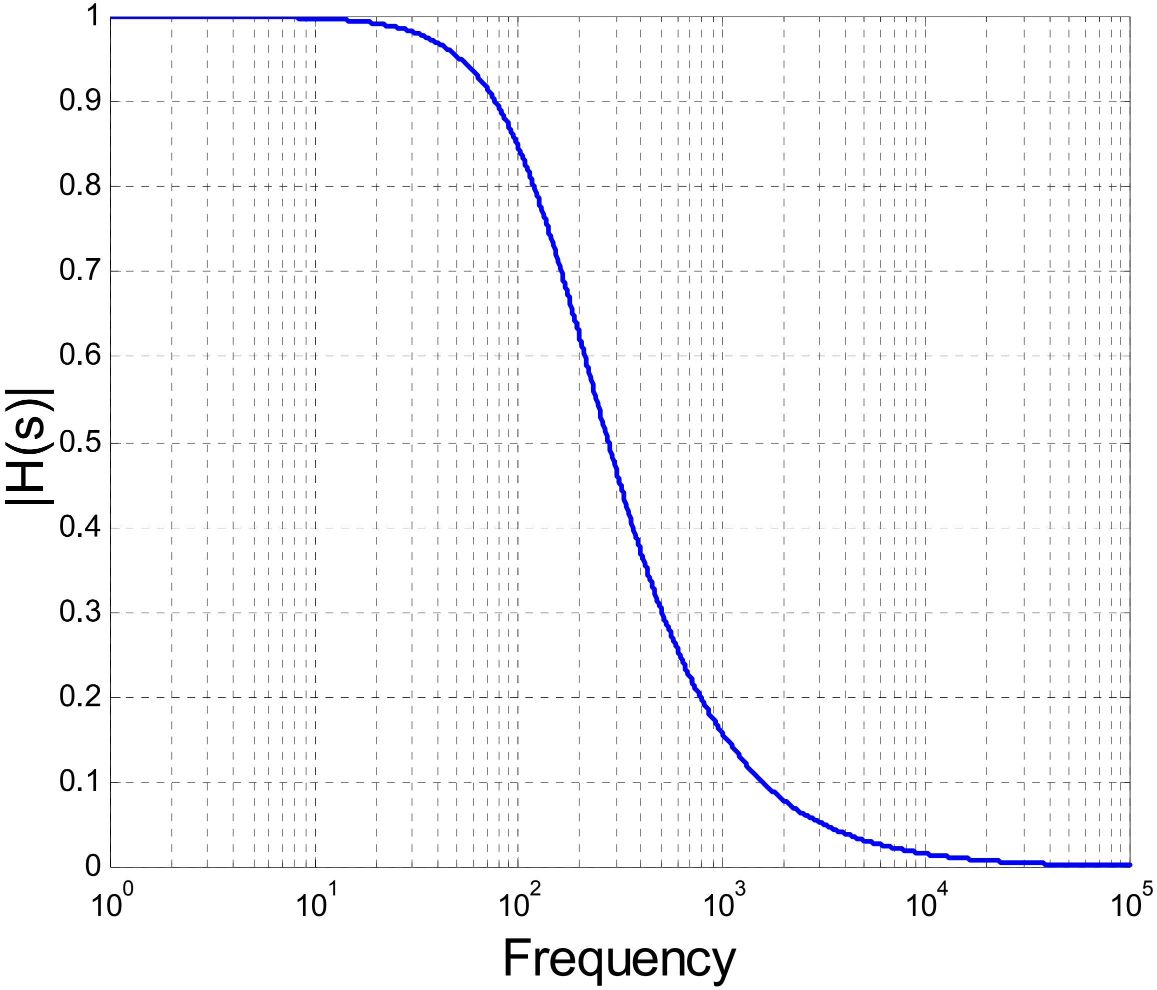
f	0	50	100	159	250	500	1000	10000	100000
ω	0	314	628	1000	1570	3141	6283	62832	628320
$\lvert\mathbf{H}(j\omega)\rvert$	1	0.95	0.85	0.707	0.54	0.3	0.16	0.016	0.0016
$\angle\mathbf{H}(j\omega)$	0	-17	-32	-45	-58	-72	-81	-89	-89.9

Amplitude Plot



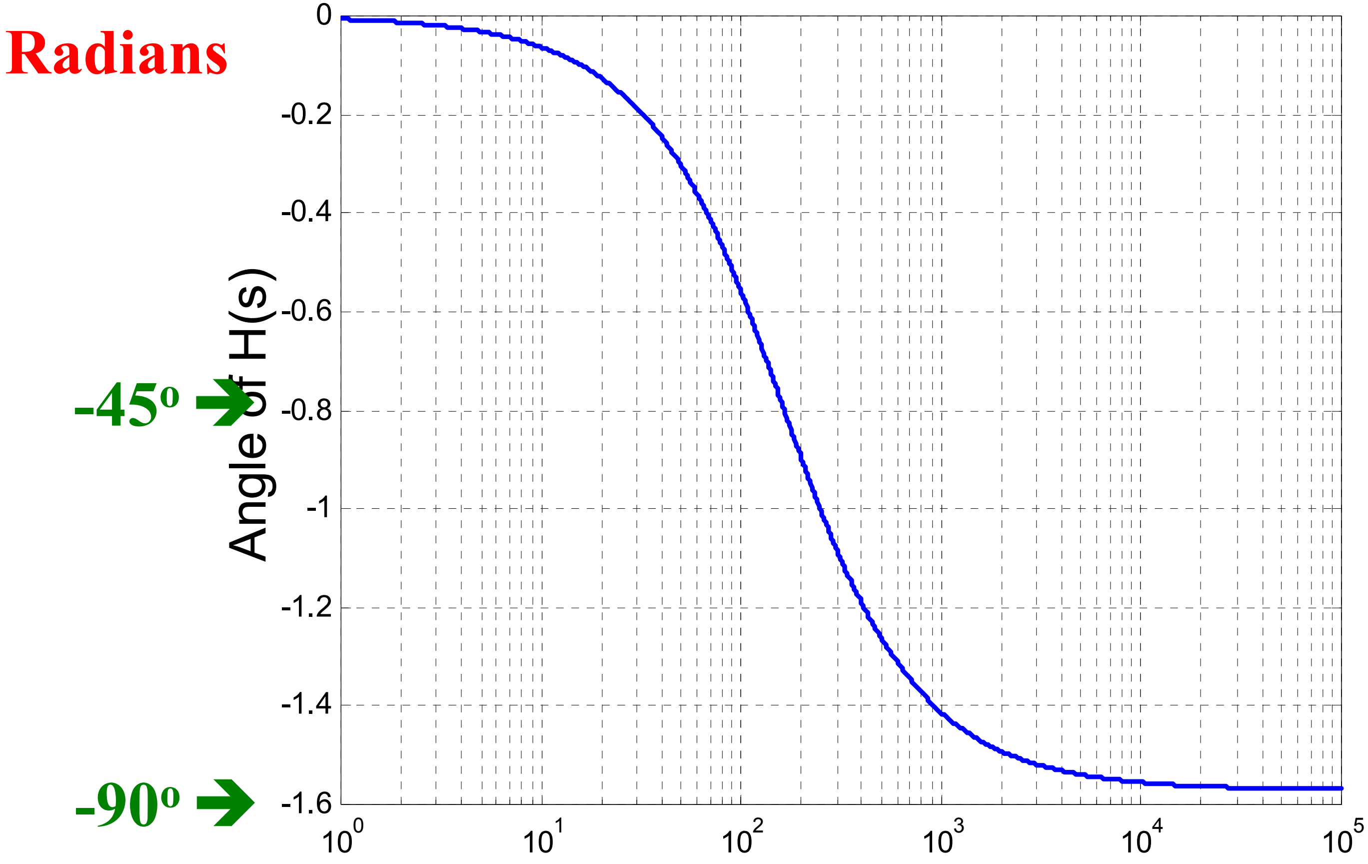
Frequency on a linear scale

Amplitude Plot



Frequency on a log scale

Phase Plot



Frequency on a log scale

Decibel Scale

→ Experimental evidence shows that the human ear perceives loudness according to a logarithmic scale.

→ The unit used to compare powers is the **decibel** which is defined as:

$$A(dB) = 10\log_{10}\left(\frac{P_1}{P_2}\right)$$

where P_1 and P_2 are the powers to be compared.

Example: Intensity of Sound

dB	Power ratio	Example
0	1	Threshold of hearing
20	10^2	Quiet room
30	10^3	Watch ticking at 1 m.
40	10^4	Quiet street
50	10^5	Quiet conversation
60	10^6	Quiet motor at 1 m.
70	10^7	Loud conversation
80	10^8	Door slamming
90	10^9	Busy typing room
100	10^{10}	Near loud motor / horn
110	10^{11}	Pneumatic drill
120	10^{12}	Near jet engine / **rock concert**
130	10^{13}	Threshold of pain

dB Scale for Voltages

Compare two voltages V_1 and V_2:

→ Choose a reference resistor R

→ The power that V_1 would dissipate in R is: $P_1 = \frac{V_1^2}{R}$

→ The power that V_2 would dissipate in R is: $P_2 = \frac{V_2^2}{R}$

$$A(dB) = 10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{V_1^2}{R}\frac{R}{V_2^2}\right)$$

$$= 10\log_{10}\left(\frac{V_1}{V_2}\right)^2 = 20\log_{10}\left(\frac{V_1}{V_2}\right)$$

dB Scale for a Transfer Function

Voltage transfer function:

$$|H(j\omega)| = \frac{|\mathbf{V_o}|}{|\mathbf{V_I}|}$$

$$A(dB) = 20\log_{10}\left(|H(j\omega)|\right)$$

Example:

$$\mathbf{H}(j\omega) = \frac{1000}{j\omega + 1000} \qquad |\mathbf{H}(j\omega)| = \frac{1000}{\sqrt{\omega^2 + 1000^2}}$$

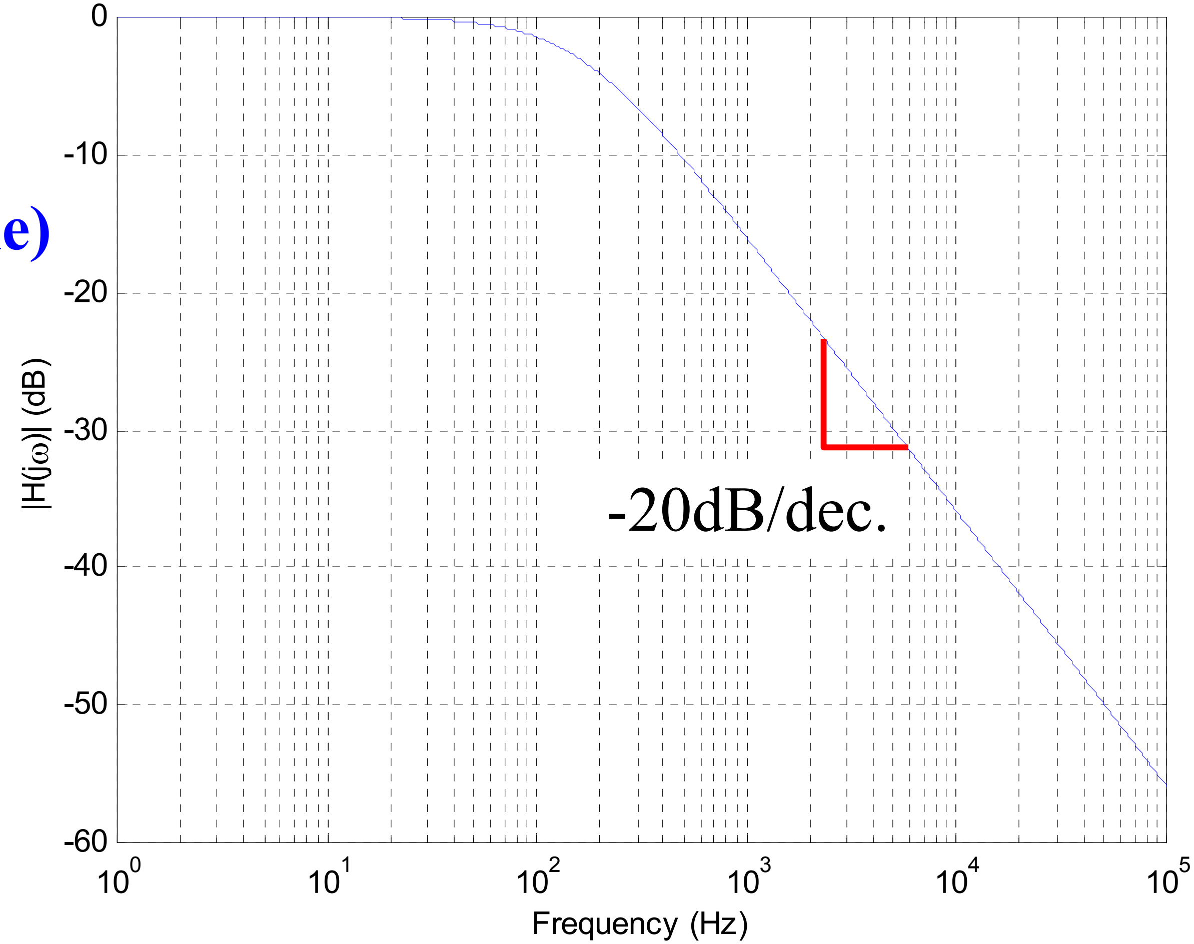
f	0	50	100	159	250	500	1000	10000	100000
ω	0	314	628	1000	1570	3141	6283	62832	628320
$\lvert\mathbf{H}(j\omega)\rvert$	1	0.95	0.85	0.707	0.54	0.3	0.16	0.016	0.0016
dB	0	−0.4	−1.4	−3	−5.3	−10.5	−15.9	−35.9	−55.9

dB Scale for a Transfer Function

$$\left|\mathbf{H}(j\omega)\right| = \frac{1000}{\sqrt{\omega^2 + 1000^2}}$$

Decibels

(linear scale)



Frequency (log scale)

dB Scale for a Transfer Function

$$\mathbf{H}(\mathbf{s}) = \frac{k}{\mathbf{s} - p} \qquad \mathbf{H}(j\omega) = \frac{k}{j\omega - p} \qquad \left|\mathbf{H}(j\omega)\right| = \frac{k}{\sqrt{\omega^2 + p^2}}$$

For ω>>p

$$\left|\mathbf{H}(j\omega)\right| = \frac{k}{\omega}$$

$$A(dB) = 20\log_{10}\left(\frac{k}{\omega}\right) = 20\log_{10}(k) - 20\log_{10}(\omega)$$

If we multiply ω by 10 → We subtract 20 from A(dB)

→ Slope is -20dB/decade.

Poles and Zeros

In general, network functions can be expressed as the ratio of two polynomials in s:

$$\mathbf{H}(\mathbf{s}) = \frac{N(\mathbf{s})}{D(\mathbf{s})} = \frac{a_m\mathbf{s}^m + a_{m-1}\mathbf{s}^{m-1} + \cdots + a_1\mathbf{s} + a_o}{b_n\mathbf{s}^n + b_{n-1}\mathbf{s}^{n-1} + \cdots + b_1\mathbf{s} + b_o}$$

where *N(***s***)* is the numerator polynomial of degree *m* and *D(***s***)* is the denominator polynomial of degree *n*

It is common to factor *N(***s***)* and *D(***s***)* to yield:

$$\mathbf{H}(\mathbf{s}) = \frac{K_o(\mathbf{s} - z_1)(\mathbf{s} - z_2)\cdots(\mathbf{s} - z_m)}{(\mathbf{s} - p_1)(\mathbf{s} - p_2)\cdots(\mathbf{s} - p_n)}$$

Poles and Zeros

$$\mathbf{H}(\mathbf{s}) = \frac{K_o(\mathbf{s}-z_1)(\mathbf{s}-z_2)\cdots(\mathbf{s}-z_m)}{(\mathbf{s}-p_1)(\mathbf{s}-p_2)\cdots(\mathbf{s}-p_n)}$$

where: K_o is a constant
z_i are the roots of N($\mathbf{s}$)
p_i are the roots of D($\mathbf{s}$)

Note: $\mathbf{s} = z_i \rightarrow \mathbf{H}(\mathbf{s})=0$

$\rightarrow z_i$ are the ***zeros*** of $\mathbf{H}(\mathbf{s})$

$\mathbf{H}(\mathbf{s}) \rightarrow \infty$ as $\mathbf{s} \rightarrow p_i$

$\rightarrow p_i$ are the ***poles*** of $\mathbf{H}(\mathbf{s})$

Note: z_i and p_i may be complex numbers

Transfer Function

$$\mathbf{H}(\mathbf{s}) = \frac{K_o(\mathbf{s}-z_1)(\mathbf{s}-z_2)\ldots(\mathbf{s}-z_m)}{(\mathbf{s}-p_1)(\mathbf{s}-p_2)\ldots(\mathbf{s}-p_n)} = \frac{K_o\prod_{i=1}^{m}(\mathbf{s}-z_i)}{\prod_{i=1}^{n}(\mathbf{s}-p_i)}$$

$$\left|\mathbf{H}(\mathbf{s})\right| = \frac{K_o\prod_{i=1}^{m}\left|(\mathbf{s}-z_i)\right|}{\prod_{i=1}^{n}\left|(\mathbf{s}-p_i)\right|}$$

$$\angle\mathbf{H}(\mathbf{s}) = \sum_{i=1}^{m}\angle(\mathbf{s}-z_i) - \sum_{i=1}^{n}\angle(\mathbf{s}-p_i)$$

$$\left|\mathbf{H}(\mathbf{s})\right|(dB) = 20\log_{10}\left(\left|\mathbf{H}(\mathbf{s})\right|\right) = 20\log_{10}\left(K_o\right) + \sum_{i=1}^{m}20\log_{10}\left(\left|(\mathbf{s}-z_i)\right|\right) - \sum_{i=1}^{n}20\log_{10}\left(\left|(\mathbf{s}-p_i)\right|\right)$$

Transfer Function

1. The amplitude and phase plots of a complex transfer function can be easily obtained by adding the factors forming the numerators and denominators.

2. Studying the behavior of first and second order circuits will allow us to handle more complex higher order circuits using the above technique.

3. The **amplitude (dB) and phase plots** on a log scale are called the Bode plots.