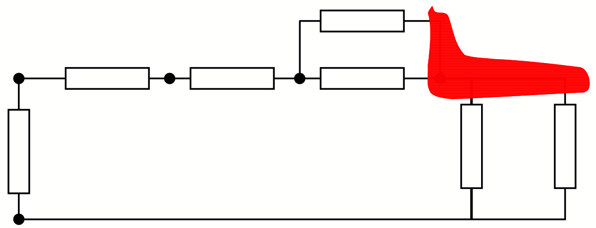
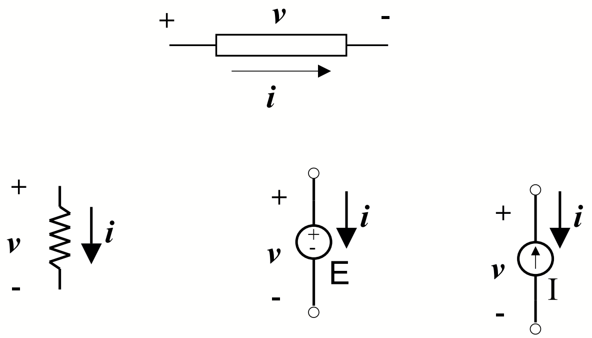
ECSE 210: Circuit Analysis

Lecture #2: Nodal Analysis

Circuits/Nodes

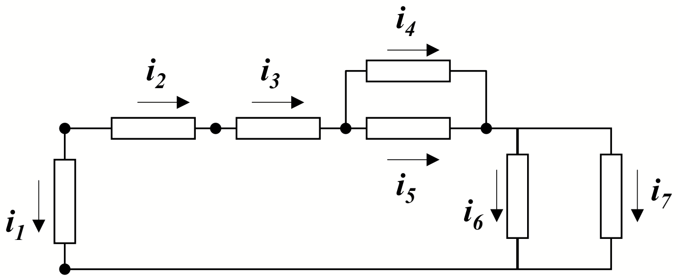


Circuit Elements



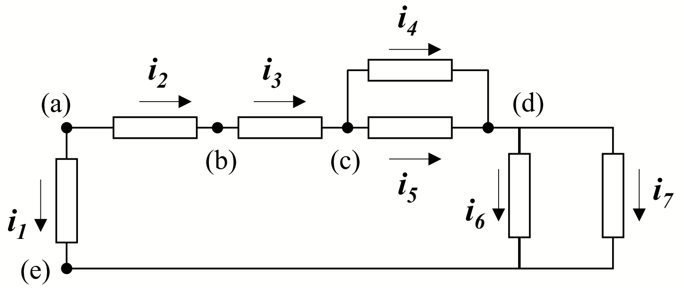
$v = iR$ $v = \mathrm{E}$ $i = -\mathrm{I}$

Circuit Variables



- Why did we chose the above current directions?
- Do we need to add voltage variables across elements?
- If we do add voltages… in what orientation?

Nodal Equations (KCL at each node)



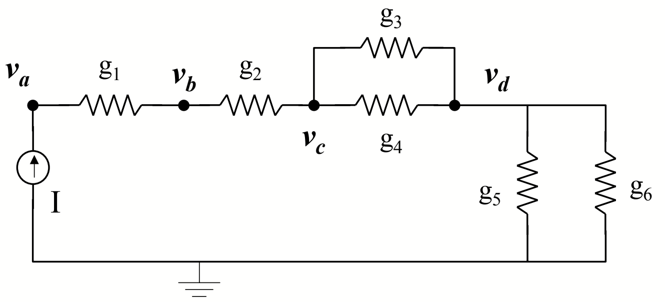
$$\left.\begin{aligned}
&(a)\quad i_1 + i_2 = 0\\
&(b)\quad -i_2 + i_3 = 0\\
&(c)\quad -i_3 + i_4 + i_5 = 0\\
&(d)\quad -i_4 - i_5 + i_6 + i_7 = 0
\end{aligned}\right\}$$

KCL equations $\sum_{out} i = 0$

5 nodes → 4 equations

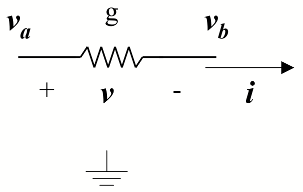
$(e)\quad -i_1 - i_6 - i_7 = 0 \longrightarrow$ Redundant!
Linear combination of first 4 eqs.

Nodal Voltages



- Choose reference node (ground).
- The voltage at the reference node is *defined* to be zero.
- Nodal voltages are defined with respect to the reference node.
- For example, v_b is the potential difference between point b and ground.

Voltage Across Elements

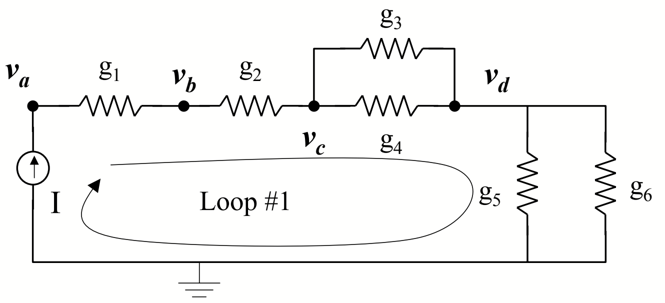


$$v = v_a - v_b$$

$$i = (v_a - v_b)\, g$$

Note: The conductance g = 1/R
Therefore, V=IR or I=gV are equivalent forms of Ohm's Law.

Kirchoff's Voltage Law (KVL)

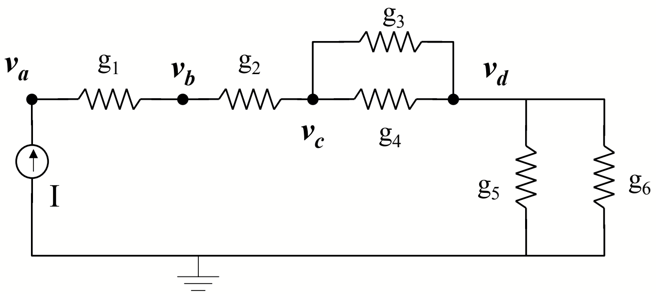


Check KVL

$$(v_a - v_b) + (v_b - v_c) + (v_c - v_d) + (v_d - 0) + (0 - v_a) = 0$$

→ Using "node voltages" means that KVL will always be satisfied.

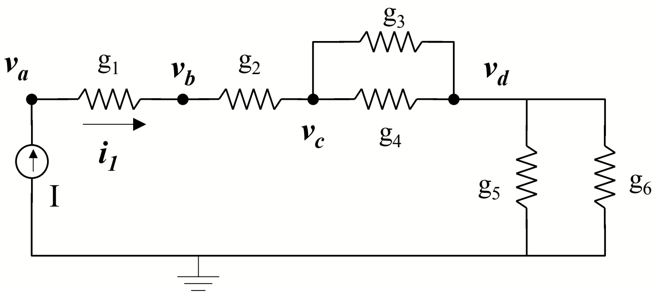
Nodal Equations



(a) $(v_a - v_b)g_1 - I = 0$ → $(v_a - v_b)g_1 = I$
(b) $(v_b - v_a)g_1 + (v_b - v_c)g_2 = 0$
(c) $(v_c - v_b)g_2 + (v_c - v_d)g_4 + (v_c - v_d)g_3 = 0$
(d) $(v_d - v_c)g_4 + (v_d - v_c)g_3 + (v_d - 0)g_5 + (v_d - 0)g_6 = 0$

4 equations in 4 unknowns → solve using linear algebra
Note: 5 nodes including ground → (5 -1) equations

Nodal Equations



Now we have solved for all voltages.

$i_1 = (v_a - v_b)g_1$

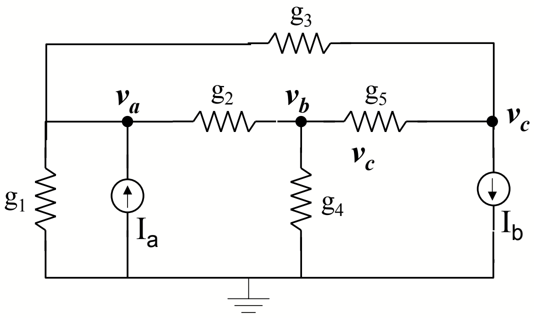
→ Similarly we can find all currents.

Nodal Analysis – Basic Steps

(1) Define all node voltage variables. Select the reference node to be used as ground.
(2) Arbitrarily define circuit branch currents.
(3) Write KCL for each node in terms of the circuit branch currents.
(4) Use Ohm's law to express branch currents in terms of node voltage variables according to the ***passive sign convention***.
(5) Solve the simultaneous algebraic equations for the unknown node voltages – *any way you like*.
(6) Determine the required characteristics from the circuit elements and node voltages.

→ See appendix A of textbook for a review of linear algebra

Nodal Analysis – Example



(a) $g_1 v_a + g_3(v_a - v_c) + g_2(v_a - v_b) = I_a$
(b) $g_2(v_b - v_a) + g_4 v_b + g_5(v_b - v_c) = 0$
(c) $g_5(v_c - v_b) + g_3(v_c - v_a) = -I_b$

Nodal Analysis – Example

$$(a)\ g_1 v_a + g_3(v_a - v_c) + g_2(v_a - v_b) = I_a$$
$$(b)\ g_2(v_b - v_a) + g_4 v_b + g_5(v_b - v_c) = 0$$
$$(c)\ g_5(v_c - v_b) + g_3(v_c - v_a) = -I_b$$

⬇

$$(a)\ (g_1 + g_3 + g_2)v_a - g_2 v_b - g_3 v_c = I_a$$
$$(b)\ -g_2\, v_a + (g_2 + g_4 + g_5)v_b - g_5 v_c = 0$$
$$(c)\ -g_3 v_a - g_5 v_b + (g_3 + g_5)v_c = -I_b$$

Nodal Analysis – Example

$$(a)\ (g_1 + g_3 + g_2)v_a - g_2 v_b - g_3 v_c = I_a$$
$$(b)\ -g_2\, v_a + (g_2 + g_4 + g_5)v_b - g_5 v_c = 0$$
$$(c)\ -g_3 v_a - g_5 v_b + (g_3 + g_5)v_c = -I_b$$

⬇

$$\begin{bmatrix} g_1 + g_2 + g_3 & -g_2 & -g_3 \\ -g_2 & g_2 + g_4 + g_5 & -g_5 \\ -g_3 & -g_5 & g_3 + g_5 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} I_a \\ 0 \\ -I_b \end{bmatrix}$$

Nodal Analysis – Example

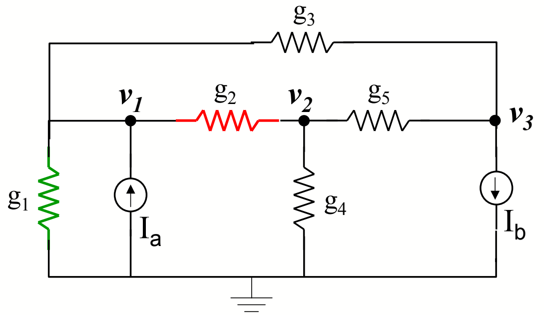
$$\begin{bmatrix} g_1 + g_2 + g_3 & -g_2 & -g_3 \\ -g_2 & g_2 + g_4 + g_5 & -g_5 \\ -g_3 & -g_5 & g_3 + g_5 \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} I_a \\ 0 \\ -I_b \end{bmatrix}$$

1. Each row represents a KCL equation.
2. Matrix is symmetric – This is not a coincidence!
3. All circuits containing resistors and current sources will have symmetric matrices.
4. Can use matrix methods to find the solution (see Appendix A of textbook)

Summary

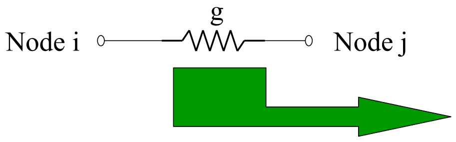
1. We have studied nodal analysis for circuits containing resistors and current sources.
2. Nodal analysis is based on applying KCL for each node in the circuit, except the reference node.
3. For a circuit containing N nodes (including ground) we therefore have N-1 equations. Each equation represents KCL at a different node.
4. Node voltages are defined with respect to the reference node (ground).
5. In this lecture, we always summed the currents ***leaving*** a node when applying KCL. Summing the currents ***entering*** a node is also valid but it is better stick to one approach.

Nodal Analysis by Inspection

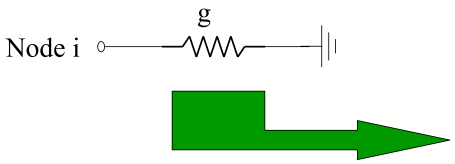


$$\begin{bmatrix} g_1+g_2+g_3 & -g_2 & -g_3 \\ -g_2 & g_2+g_4+g_5 & -g_5 \\ -g_3 & -g_5 & g_3+g_5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} I_a \\ 0 \\ -I_b \end{bmatrix}$$

Nodal Analysis by Inspection

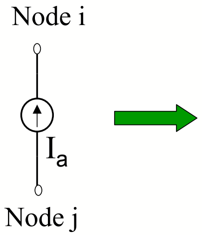


$$\begin{array}{c|cc} & \text{col i} & \text{col j} \\ \hline \text{row i} & g & -g \\ \text{row j} & -g & g \end{array}$$



$$\begin{array}{c|c} & \text{col i} \\ \hline \text{row i} & g \end{array}$$

Nodal Analysis by Inspection



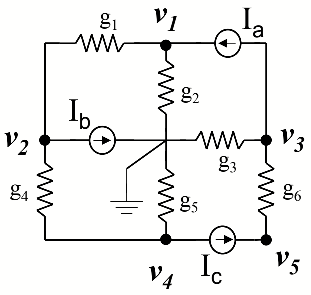
$$\begin{array}{c|c} & \text{RHS vector} \\ \hline \text{row i} & I_a \\ \text{row j} & -I_a \end{array}$$

1. Be careful about the signs.
2. If one of the terminals is connected to ground, then there is no corresponding KCL equation and that node simply does not appear in the equations (we get only one entry in the RHS vector).

Nodal Analysis by Inspection

1. Designate a reference node.
2. Number the remaining nodes.
3. Size of the matrix (number of eqs.) is equal to the number of nodes (not including the reference node).
4. Add the contribution of each element to the matrix equations.
5. Resistors contribute to the LHS equations or to the matrix itself.
6. Current sources contribute to the right hand side (RHS) vector.

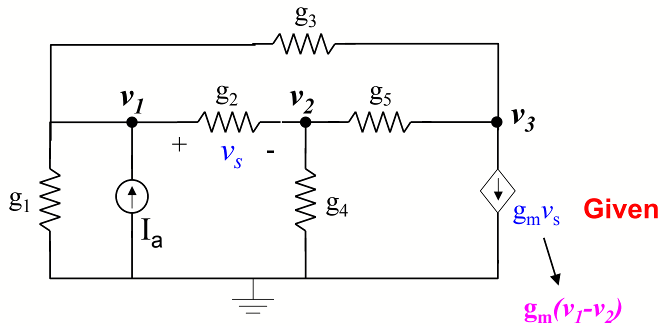
Nodal Analysis by Inspection: Example



$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & g_1+g_2 & -g_1 & 0 & 0 & 0 \\ 2 & -g_1 & g_1+g_4 & 0 & -g_4 & 0 \\ 3 & 0 & 0 & g_3+g_6 & 0 & -g_6 \\ 4 & 0 & -g_4 & 0 & g_4+g_5 & 0 \\ 5 & 0 & 0 & -g_6 & 0 & g_6 \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{array}{cc} \left[\begin{matrix} I_a \\ -I_b \\ -I_a \\ -I_c \\ I_c \end{matrix}\right] & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

→ Check to see that each row represents the KCL equation at the corresponding node.

Dependent Current Source



KCL at node 1: $v_1 g_1 + (v_1 - v_3) g_3 + (v_1 - v_2) g_2 = I_a$

KCL at node 2: $v_2 g_4 + (v_2 - v_1) g_2 + (v_2 - v_3) g_5 = 0$

KCL at node 3: $(v_3 - v_2) g_5 + (v_3 - v_1) g_3 + \mathbf{(v_1 - v_2) g_m} = 0$

Dependent Current Source

KCL at node 1: $v_1 g_1 + (v_1 - v_3) g_3 + (v_1 - v_2) g_2 = I_a$

KCL at node 2: $v_2 g_4 + (v_2 - v_1) g_2 + (v_2 - v_3) g_5 = 0$

KCL at node 3: $(v_3 - v_2) g_5 + (v_3 - v_1) g_3 + (v_1 - v_2) g_m = 0$

$$\begin{bmatrix} g_1+g_2+g_3 & -g_2 & -g_3 \\ -g_2 & g_2+g_4+g_5 & -g_5 \\ -g_3+g_m & -g_5-g_m & g_3+g_5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} I_a \\ 0 \\ 0 \end{bmatrix}$$

Dependent sources can destroy symmetry.