ECSE 210: Circuit Analysis

Lecture #15:

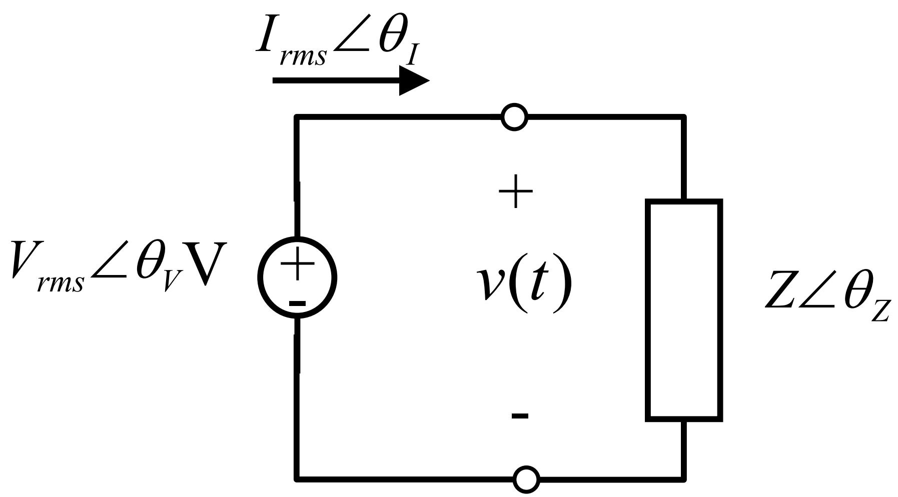
Steady State Power Analysis

AC Steady-State Power

Recall:

$$V_{rms}\angle\theta_V = \left(I_{rms}\angle\theta_I\right)\left(Z\angle\theta_Z\right) = I_{rms}Z\angle(\theta_I+\theta_Z)$$

$$\theta_Z = \theta_V - \theta_I$$



$$P_{avg} = I_{rms}V_{rms}\cos(\theta_V-\theta_I)$$

$$P_{avg} = I_{rms}V_{rms}\cos(\theta_Z)$$

$$P_{avg} = I_{rms}^2 Z\cos(\theta_Z) \qquad P_{avg} = \frac{V_{rms}^2}{Z}\underbrace{\cos(\theta_Z)}_{\text{power factor}}$$

power factor

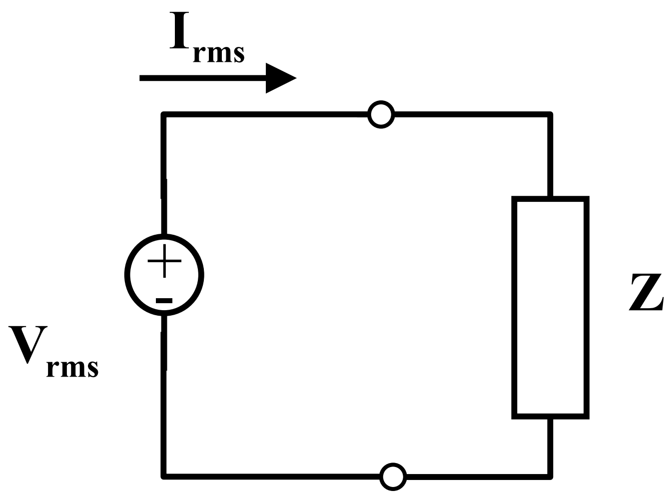
$$power\ \ factor = \cos(\theta_Z) = \cos(\theta_V-\theta_I)$$

Complex Power

$\mathbf{V_{rms}} = V_{rms}\angle\theta_V$ $\mathbf{Z} = Z\angle\theta_Z$

$\mathbf{I_{rms}} = I_{rms}\angle\theta_I$

$\mathbf{I^*_{rms}} = I_{rms}\angle -\theta_I$



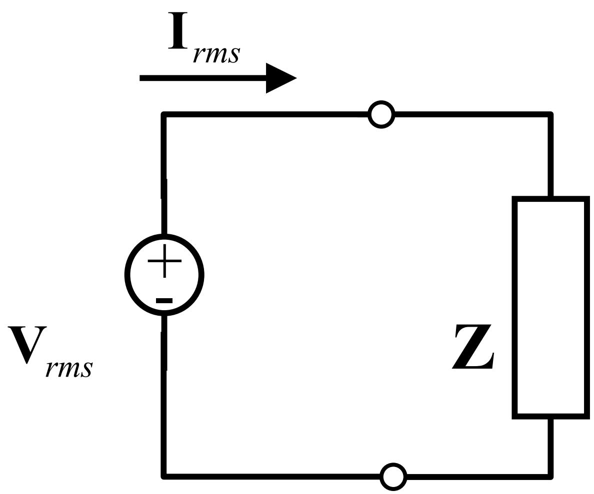
Define complex power as:

$$\mathbf{S} = \mathbf{V_{rms}}\mathbf{I^*_{rms}}$$

$$\mathbf{S} = \mathbf{V_{rms}}\mathbf{I^*_{rms}} = V_{rms}I_{rms}\angle(\theta_v - \theta_I) = V_{rms}I_{rms}e^{j(\theta_v - \theta_I)}$$

$$= \underbrace{V_{rms}I_{rms}\cos(\theta_v - \theta_I)}_{P_{avg}} + jV_{rms}I_{rms}\sin(\theta_v - \theta_I)$$

AC Steady-State Power



Complex power:

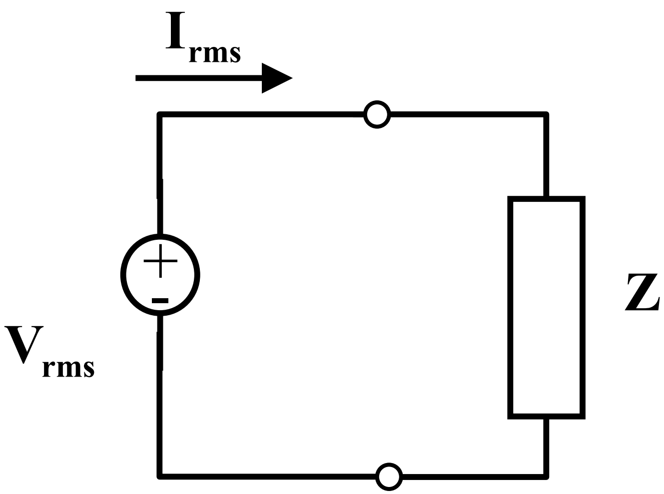
$$\mathbf{S} = \mathbf{V_{rms}}\mathbf{I^*_{rms}} \Rightarrow P_{avg} = \mathrm{Re}[\mathbf{S}]$$

$$\mathbf{V_{rms}} = \mathbf{Z}\mathbf{I_{rms}}$$

$$\mathbf{S} = \mathbf{Z}\mathbf{I_{rms}}\mathbf{I^*_{rms}} = \mathbf{Z}\left|\mathbf{I_{rms}}\right|^2 = \mathbf{Z}I^2_{rms}$$

$$P_{avg} = \mathrm{Re}[\mathbf{S}] = \mathrm{Re}[\mathbf{Z}]I^2_{rms} = I^2_{rms}Z\cos(\theta_z)$$

Nomenclature/Conventions



$$\mathbf{S} = \mathbf{V}_{\mathbf{rms}}\mathbf{I}^{*}_{\mathbf{rms}} = V_{rms} I_{rms} \angle(\theta_V - \theta_I)$$

$$\theta_V - \theta_I = \theta_Z$$

$$\mathbf{S} = P + jQ$$

→ **S** is a complex number.

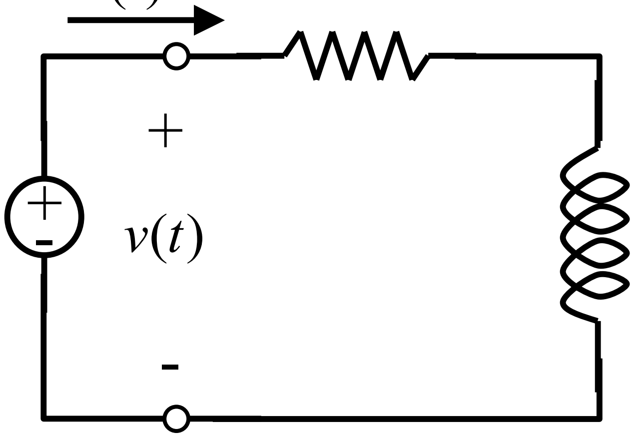
→ The magnitude of **S** is the ***apparent power***.
(Units: volt-ampere VA or kVA)

→ The angle of **S** is the ***power factor angle***.

→ The real part of **S** is the ***average power***. **(Units: watts)**

→ The imaginary part of **S** is the ***reactive power***.
(Units: volt-ampere reactive (VAR))

Example



$$\mathbf{V}_{rms} = \frac{10}{\sqrt{2}}\angle 0^o = 7.07\angle 0^o \text{ V}$$

$$\mathbf{Z} = 50 + j10^4(12\times 10^{-3}) = 50 + j120\Omega$$

$$\mathbf{I}_{rms} = \frac{\mathbf{V}_{rms}}{\mathbf{Z}}$$

$$\mathbf{I}_{rms} = \frac{7.07\angle 0^o}{130\angle 67.38^o} = 54.4\angle -67.38^o \text{ mA}$$

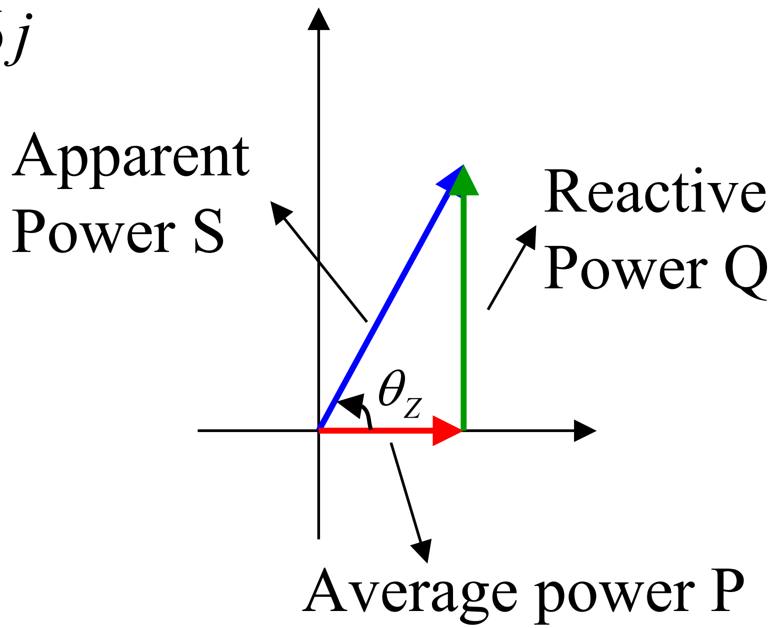
$$\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = (7.07\angle 0^o V)(54.4\angle 67.38^o mA) = 0.385\angle 67.38^o \text{ VA}$$

$$\mathbf{S} = \mathbf{V}_{rms}\mathbf{I}^*_{rms} = 0.385\cos(67.38^o) + j0.385\sin(67.38^o) = 0.15 + 0.36j$$

Example

$$\mathbf{S} = \mathbf{V_{rms}}\mathbf{I}^*_{\mathbf{rms}} = 0.385\angle 67.38^o = 0.15 + 0.36j$$

Apparent power	$S = 0.385$ VA
Average power	$P=0.15$ W
Reactive power	$Q=0.36$ VAR
Power factor angle	$= 67.38^o$



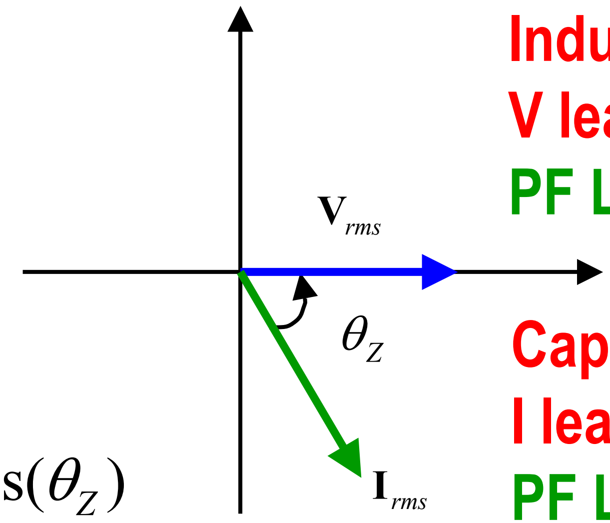
Power triangle

Power Factor

$$P_{avg} = I_{rms}V_{rms}\cos(\theta_V - \theta_I) = I_{rms}V_{rms}\cos(\theta_Z)$$

$$power\ \ factor = \frac{Average\ power}{Apparent\ power}$$

$$power\ \ factor = \frac{P_{avg}}{I_{rms}V_{rms}} = \cos(\theta_V - \theta_I) = \cos(\theta_Z)$$



Inductive
V leads I
PF Lagging

Capacitive
I leads V
PF Leading

→ For an inductive load, the current lags the voltage and we have a **lagging power factor**.

→ For a capacitive load the current leads the voltage and we have a **leading power factor**.

→ **It is the current that lags and leads not the power factor!**

→ For a purely resistive load the power factor is 1.

→ For a purely reactive load the power factor is 0.

Power Factor

Case Study:

An industrial load consumes 50kW at a pf of 0.8 lagging, from a 220V (rms), 60Hz line.

What can we conclude?

➔ Lagging pf → Current is lagging.

➔ $$P_{avg} = I_{rms}V_{rms} \times pf = I_{rms}(220V)(0.8) = 50000W$$

➔ $$I_{rms} = \frac{50000W}{(220V)(0.8)} = 284.1A$$

➔ $$\cos(\theta_V - \theta_I) = 0.8 \quad \rightarrow \quad |\theta_V - \theta_I| = 36.9^o$$

Power Factor

Given: $\mathbf{V}_{\mathbf{rms}} = 220\angle 0^o V$ **and** **lagging** current

➔ $\mathbf{I}_{\mathbf{rms}} = 284.1\angle - 36.9^o A$ and $|\theta_V - \theta_I| = 36.9^o$

V leads I

➔ The impedance of the load is:

$$\mathbf{Z}_L = \frac{\mathbf{V}_{\mathbf{rms}}}{\mathbf{I}_{\mathbf{rms}}} = \frac{220}{284.1\angle - 36.9^o} = 0.774\angle 36.9^o \Omega$$

Inductive Impedance

Power Factor

$$\mathbf{V_{rms}} = 220\angle 0^o V \qquad \mathbf{I_{rms}} = 284.1\angle -36.9^o A \qquad P_{avg} = 50kW$$

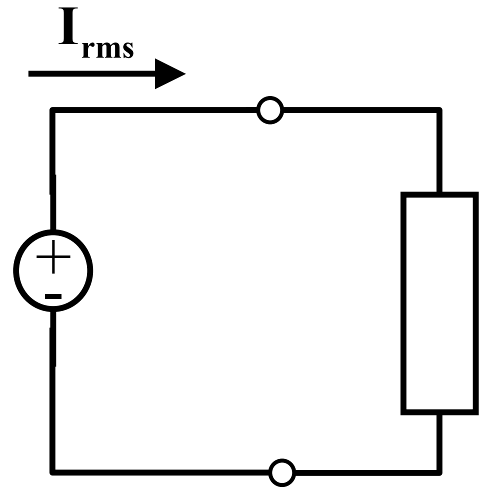
- ➔ **What is the problem with the situation above?**
- ➔ How much current is required to supply the same power to a purely resistive load (pf=1) ?

$$P_{avg} = 50kW = V_{rms} I_{rms} = 220 I_{rms} \quad \rightarrow \quad I_{rms} = 227.3A$$

- ➔ **A resistive load requires much less current.**
- ➔ This means less resistive losses (I^2R) in the transmission lines.
- ➔ The large current in the reactive case (e.g., pf=0.8) results from the transfer of energy back and forth from the source to the load and vice versa.

Power Factor Correction

- Suppose we wish to raise the power factor of the plant from 0.8 lagging to 0.95 lagging.



- $\mathbf{Z} = 0.774\angle 36.9^o \Omega$
- $\mathbf{Y} = \frac{1}{\mathbf{Z}} = 1.29\angle -36.9^o S$

$$\mathbf{Y} = 1.03 - j0.775S$$

- Existing pf:

$$pf = \cos(36.9^o) = 0.8$$

- Desired pf:

$$pf = \cos(\theta^o_{new}) = 0.95$$

→ **Add parallel capacitor**

Power Factor Correction

$$\mathbf{Y}_{new} = 1.03 + j(B_c - 0.775)$$

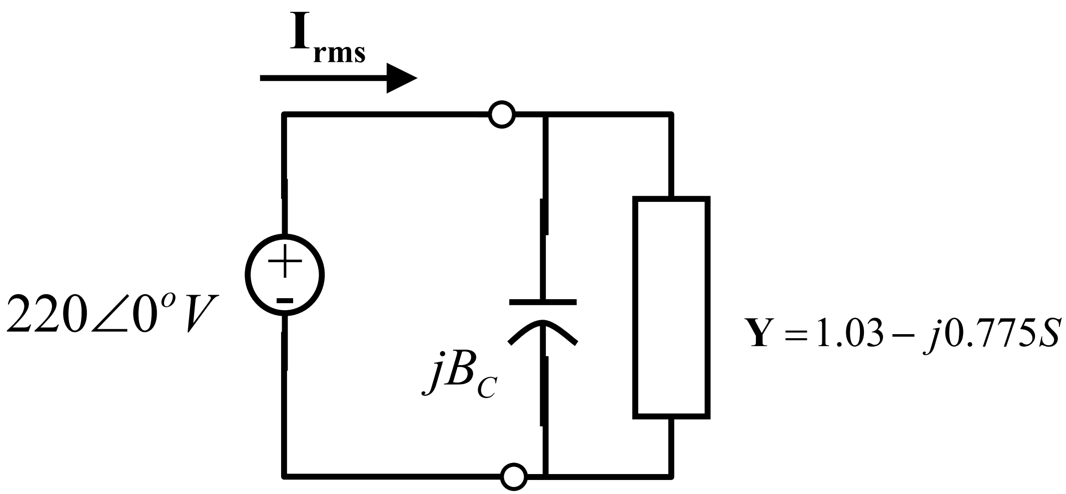
$$pf_{new} = \cos(\theta^o_{new}) = 0.95$$

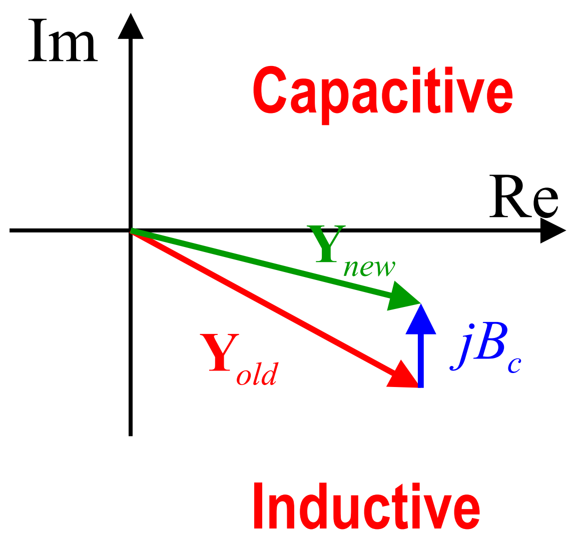
$$\theta^o_{new} = -18.19^o$$

$$\tan(\theta^o_{new}) = \frac{B_c - 0.775}{1.03} = -0.323$$

$$B_c = 0.437S = \omega C = (2\pi \times 60)C$$

$$C = 1158\mu F$$

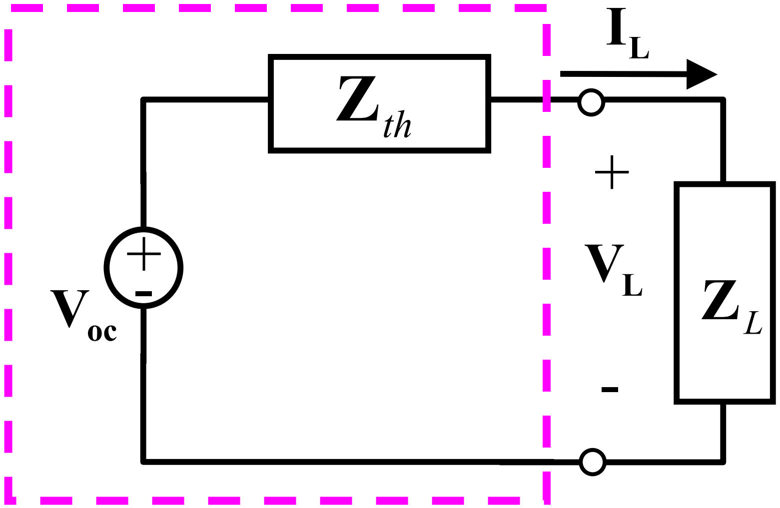




Maximum Average Power Transfer

Recall:

For resistive circuits, maximum power transfer occurs when the load R_L is set to the **Thevenin equivalent resistance of the remainder of the circuit**.

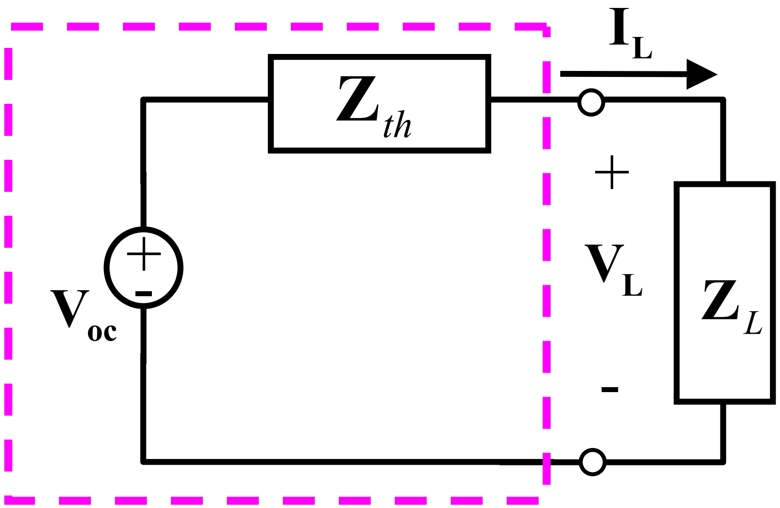


→ What is the relation between $\mathbf{Z}_{th}$ and $\mathbf{Z}_L$ which yields the *maximum average power transfer* to the load $\mathbf{Z}_L$?

→ In the above circuit, $\mathbf{V}_{oc}$ is the RMS source voltage phasor, $\mathbf{V}_L$ is the RMS voltage phasor at the load, and $\mathbf{I}_L$ is the RMS current phasor.

Maximum Average Power Transfer

$$\mathbf{Z}_{th} = R_{th} + jX_{th} \qquad \mathbf{Z}_L = R_L + jX_L$$



$$\mathbf{V_{oc}} = V_{oc}\angle\theta_{oc}$$

$$\mathbf{I_L} = \frac{\mathbf{V}_{oc}}{\mathbf{Z}_{th} + \mathbf{Z}_L}$$

$$\mathbf{V}_L = \frac{\mathbf{Z}_L}{\mathbf{Z}_{th} + \mathbf{Z}_L}\mathbf{V}_{oc}$$

$$\mathbf{S} = \mathbf{V_L I_L^*} = \frac{\mathbf{Z}_L}{(\mathbf{Z}_{th} + \mathbf{Z}_L)} \frac{1}{(\mathbf{Z}_{th} + \mathbf{Z}_L)^*}\mathbf{V}_{oc}\mathbf{V}_{oc}^*$$

Maximum Average Power Transfer

$$\mathbf{Z}_{th} = R_{th} + jX_{th} \qquad \mathbf{Z}_L = R_L + jX_L \qquad \mathbf{V_{oc}} = V_{oc}\angle\theta_{oc}$$

$$\mathbf{S} = \mathbf{V_L I_L^*} = \frac{\mathbf{Z}_L}{(\mathbf{Z}_{th} + \mathbf{Z}_L)} \frac{1}{(\mathbf{Z}_{th} + \mathbf{Z}_L)^*} \mathbf{V}_{oc}\mathbf{V}_{oc}^*$$

$$\mathbf{V_{oc}V_{oc}^*} = V_{oc}^2$$

$$\mathbf{Z}_{th} + \mathbf{Z}_L = R_{th} + R_L + j(X_{th} + X_L)$$

$$(\mathbf{Z}_{th} + \mathbf{Z}_L)(\mathbf{Z}_{th} + \mathbf{Z}_L)^* = (R_{th} + R_L)^2 + (X_{th} + X_L)^2$$

$$\mathbf{S} = \frac{R_L + jX_L}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2} V_{oc}^2$$

$$P_{avg} = \mathrm{Re}[\mathbf{S}] = \frac{R_L V_{oc}^2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

Maximum Average Power Transfer

$$P_{avg} = \mathrm{Re}[\mathbf{S}] = \frac{R_L V_{oc}^2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

Maximize the power: $X_L = -X_{th}$

$$P_{avg} = \mathrm{Re}[\mathbf{S}] = \frac{R_L V_{oc}^2}{(R_{th} + R_L)^2}$$

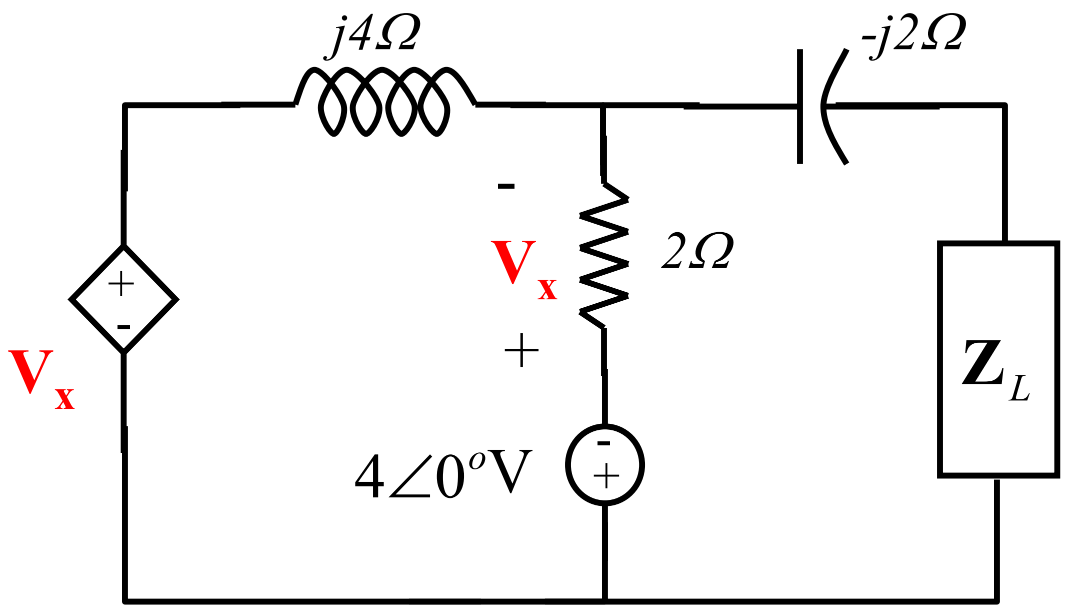
$$\frac{dP_{avg}}{dR_L} = \left(\frac{1}{(R_{th} + R_L)^2} - \frac{2R_L}{(R_{th} + R_L)^3} \right) V_{oc}^2 = \frac{R_{th} - R_L}{(R_{th} + R_L)^3} V_{oc}^2$$

Maximum power occurs at: $R_L = R_{th}$

$$\mathbf{Z}_L = R_{th} - jX_{th} = \mathbf{Z}_{th}^*$$

Example

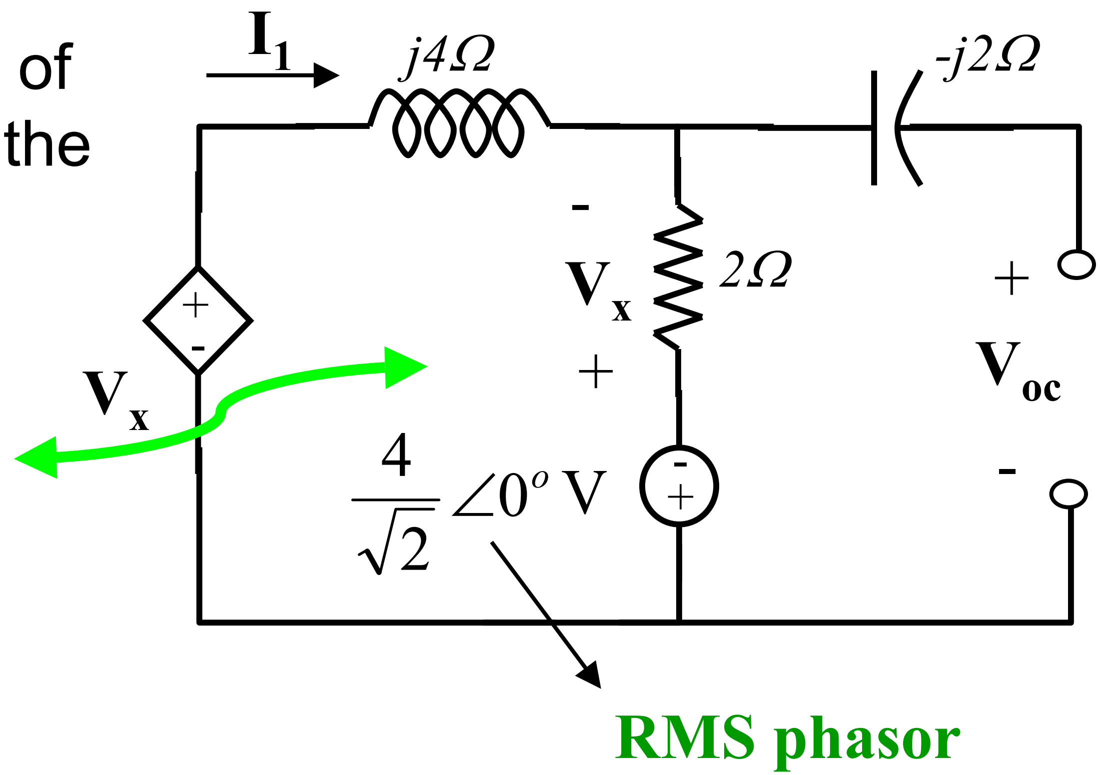
Find the value of $\mathbf{Z}_L$ for maximum average power transfer. Then find the average power delivered to $\mathbf{Z}_L$.



→ First find the Thevenin equivalent of the circuit seen from the terminals of $\mathbf{Z}_L$.

Example

→ Thevenin equivalent of the circuit seen from the terminals of $\mathbf{Z}_L$.



KVL

$$-\mathbf{V}_x + j4\mathbf{I}_1 - \mathbf{V}_x - \frac{4}{\sqrt{2}} = 0$$

Also: $\mathbf{V}_x = -2\mathbf{I}_1$

$$-2\mathbf{V}_x - j2\mathbf{V}_x = \frac{4}{\sqrt{2}}$$

$$\mathbf{V}_x = 1\angle 135^o V$$

$$\mathbf{V}_{oc} = -\left(1\angle 135^o + \frac{4}{\sqrt{2}}\right) = -2.24\angle 18.43^o V = \frac{-3-j}{\sqrt{2}} \quad \textbf{V rms}$$

Example

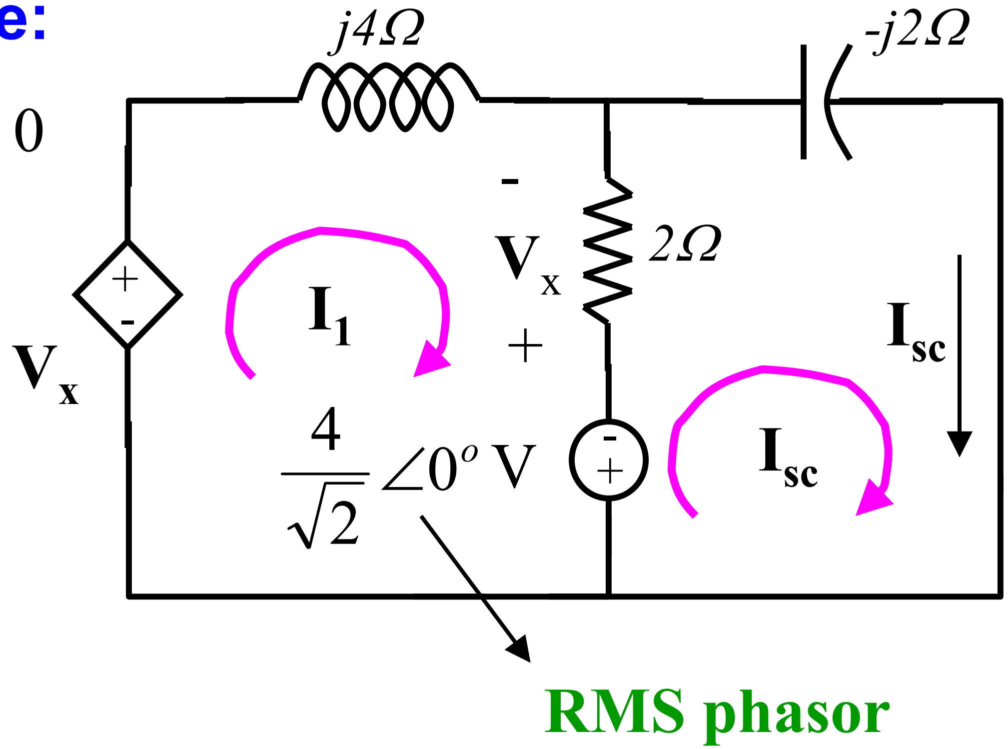
Find Thevenin impedance:

$$-\mathbf{V_x} + j4\mathbf{I_1} + 2(\mathbf{I_1} - \mathbf{I_{sc}}) - \frac{4}{\sqrt{2}} = 0$$

$$\frac{4}{\sqrt{2}} + 2(\mathbf{I_{sc}} - \mathbf{I_1}) - j2\mathbf{I_{sc}} = 0$$

$$\mathbf{V_x} = -2(\mathbf{I_1} - \mathbf{I_{sc}})$$

➔ $$\mathbf{I_{sc}} = -\frac{(1+2j)}{\sqrt{2}}$$

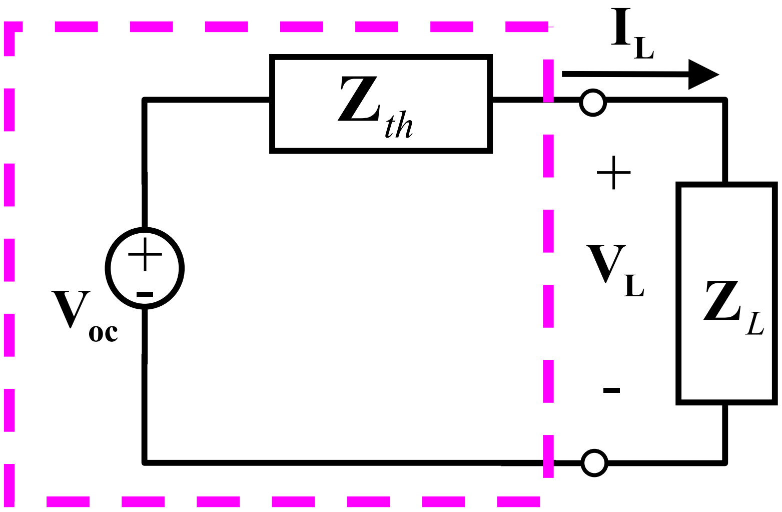


$$\mathbf{Z}_{th} = \frac{\mathbf{V_{oc}}}{\mathbf{I_{sc}}} = \frac{\frac{3+j}{\sqrt{2}}}{\frac{(1+2j)}{\sqrt{2}}} = 1 - j\Omega$$

Example

$$\mathbf{V_{oc}} = -2.24\angle 18.43^o V \quad \text{rms}$$

$$\mathbf{Z}_{th} = 1 - j = \sqrt{2}\angle -45^o \Omega$$



For maximum average power transfer:

$$\mathbf{Z}_L = \mathbf{Z}_{th}^* = 1 + j = \sqrt{2}\angle 45^o \Omega$$

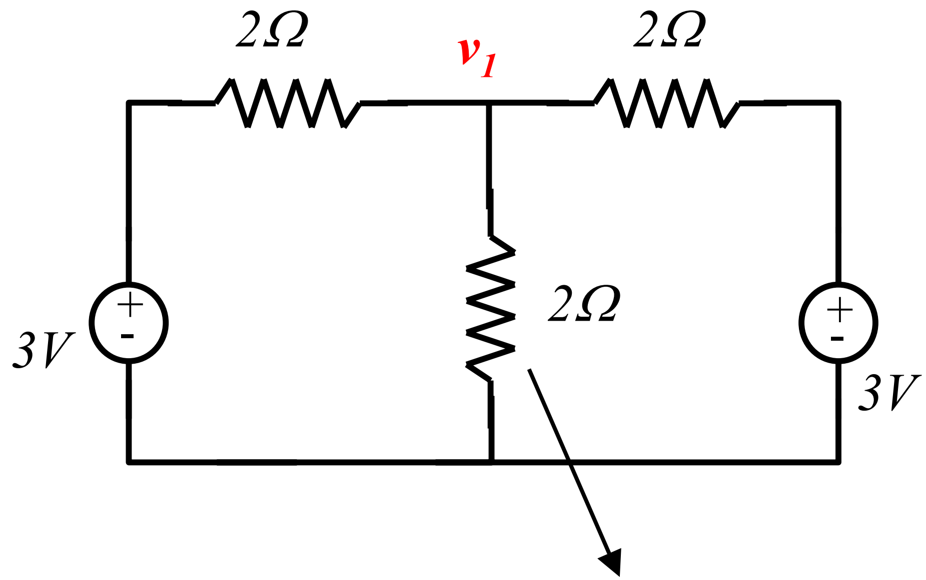
$$\mathbf{I_L} = \frac{-2.24\angle 18.43}{2} = -1.118\angle 18.43^o \quad \text{A}$$

$$\mathbf{V_L} = \mathbf{I_L}\mathbf{Z}_L \qquad \mathbf{S} = \mathbf{I_L^*}\mathbf{V_L} = \mathbf{I_L}\mathbf{I_L^*}\mathbf{Z}_L = (1.118)^2(1 + j)$$

$$P_{avg} = \text{Re}[\mathbf{S}] = (1.118)^2 = 1.25W$$

Superposition and Power

Consider the following dc circuit:



We can use superposition to find v_1.

☆ v_1 due to first source alone:

$$v_1' = \frac{2\Omega \,||\, 2\Omega}{2\Omega \,||\, 2\Omega + 2\Omega} 3 = 1V$$

🕐 v_1 due to second source alone:

$$v_1'' = \frac{2\Omega \,||\, 2\Omega}{2\Omega \,||\, 2\Omega + 2\Omega} 3 = 1V$$

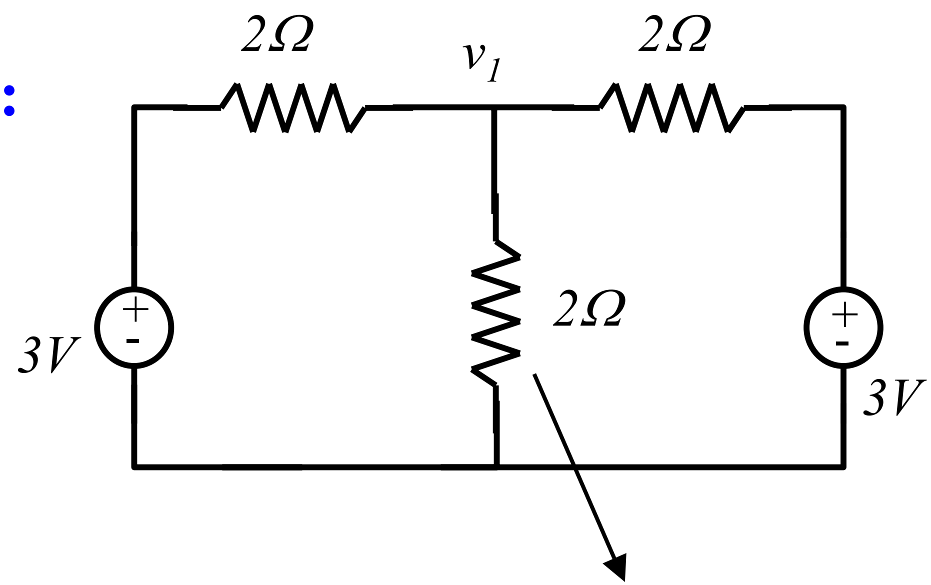
Total voltage: $v_1 = v_1' + v_1'' = 2V$

Calculate power dissipated in this resistor.

Power: $P = \frac{v_1^2}{R} = 2W$

Superposition and Power

Consider the same dc circuit again:



Try superposition directly on power.

Calculate power dissipated in this resistor.

Power due to first source alone

$$P_1 = \frac{(1)^2}{2} = 0.5W \qquad V_1 = 1$$

Power due to second source alone

$$P_2 = \frac{(1)^2}{2} = 0.5W \qquad V_1 = 1$$

Total power: $P = P_1 + P_2 = 1W$ → **Wrong answer!!!**

Superposition and Power

1. In general we cannot apply the superposition principle directly to determine the power.

2. One can show that **power superposition** works for sources of **different frequencies**, but we will not worry about this here.

3. **Do not apply superposition directly to calculate the power!**