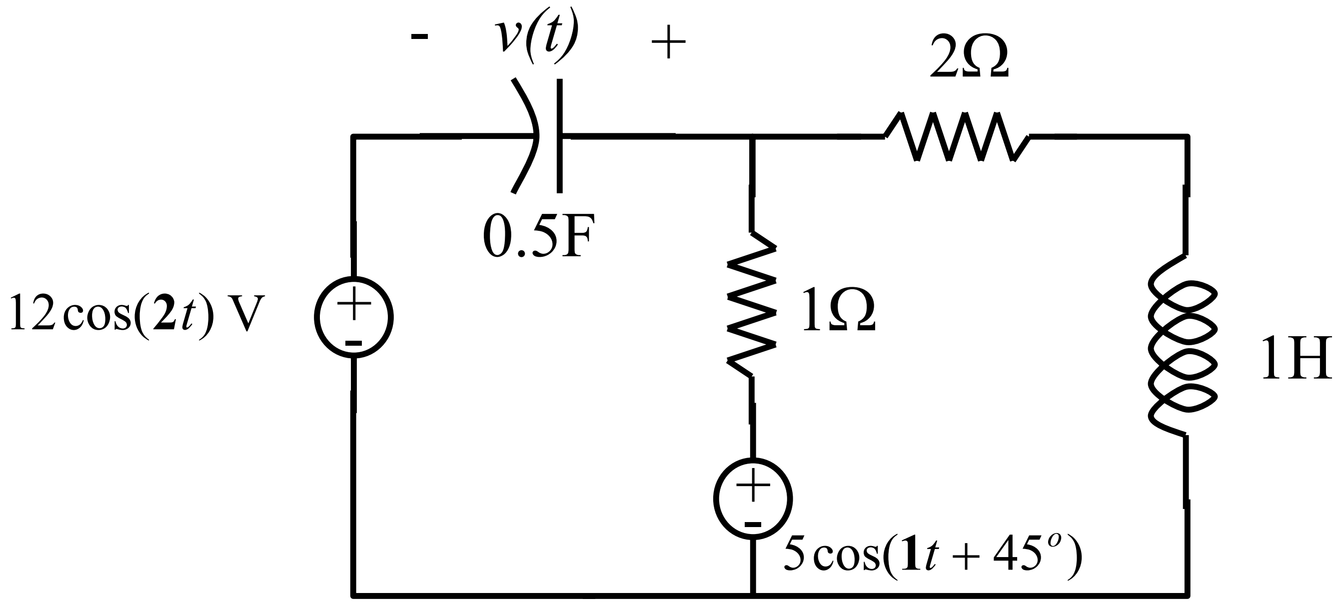
ECSE 210: Circuit Analysis

Lecture #14:

AC Steady State Analysis

Steady State Power Analysis

Sources With Different Frequencies

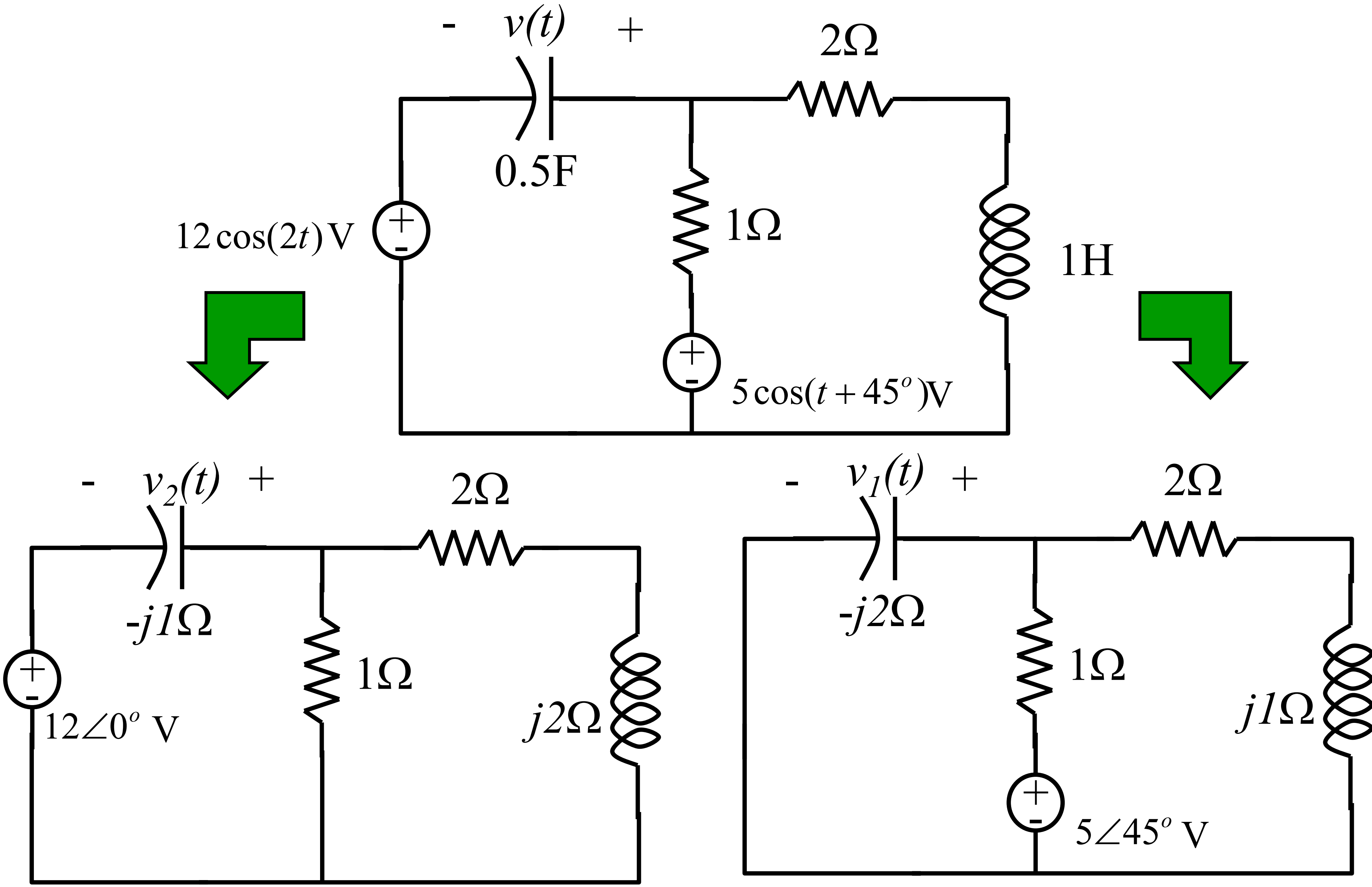


→ Two sources at different frequencies.

→ We must use superposition.

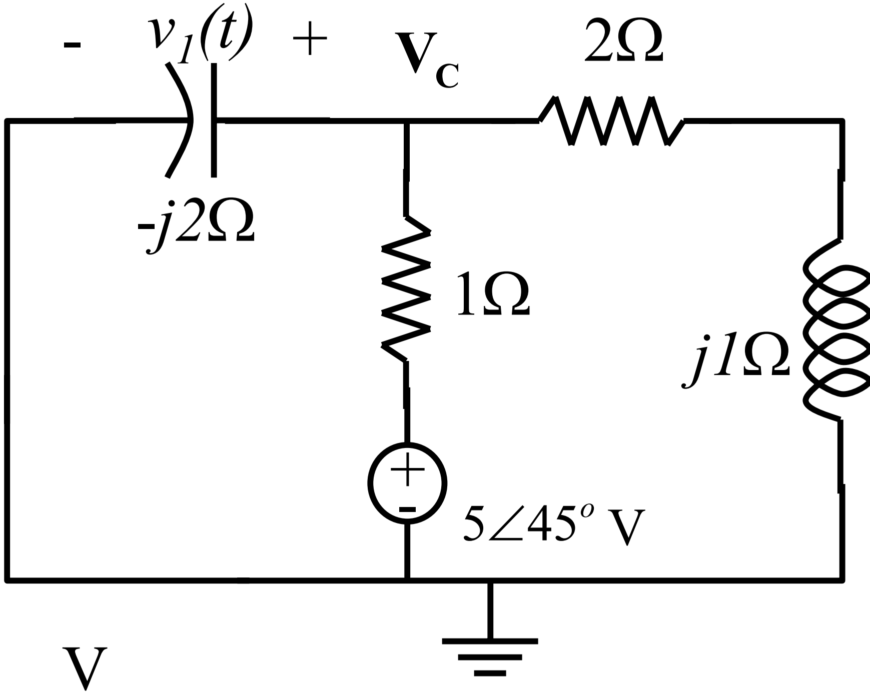
(Only meaningful in the time domain!)

Sources With Different Frequencies



Sources With Different Frequencies

$$\mathbf{V_C} = \frac{(-j2\Omega)\,||\,(2\Omega + j1\Omega)}{1\Omega + (-j2\Omega)\,||\,(2\Omega + j1\Omega)}(5\angle 45^o)$$



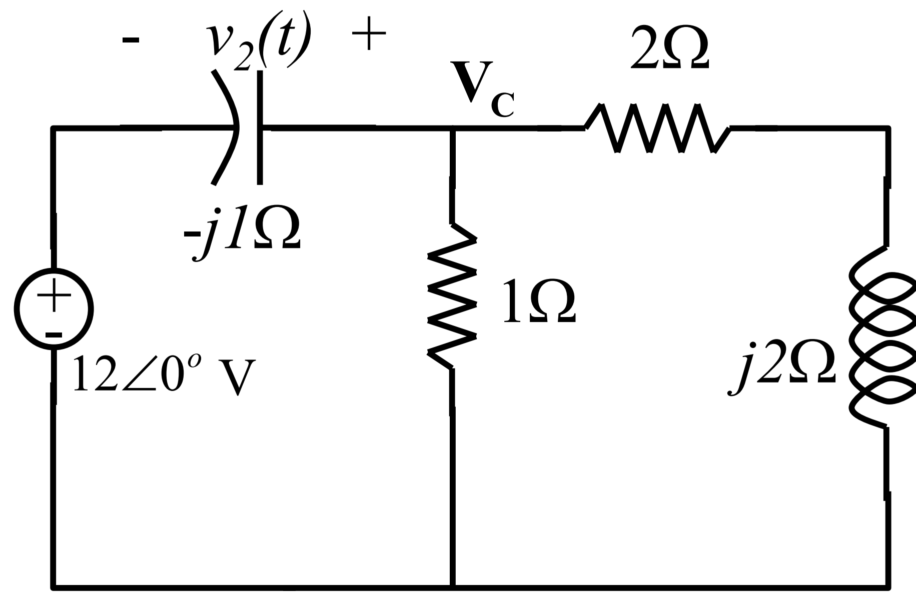
V

$$\mathbf{V_C} = (0.7\angle -12.09^o)(5\angle 45^o) = 3.5\angle 32.9^o$$

$$v_1(t) = 3.5\cos(t + 32.9^o)\ \text{V}$$

Sources With Different Frequencies

$$\mathbf{V_C} = \frac{(2\Omega + j2\Omega) \,||\, (1\Omega)}{(2\Omega + j2\Omega) \,||\, (1\Omega) - j1\Omega}(12\angle 0^o)$$



$$\mathbf{V_C} = (0.686\angle 59.04^o)(12\angle 0^o) = 8.23\angle 59.04^o \ \text{V}$$

$$\mathbf{V_2} = (8.23\angle 59.04^o) - (12\angle 0^o) = 10.5\angle 138^o \ \text{V}$$

$$v_2(t) = 10.5\cos(2t + 138^o) \ \text{V}$$

Sources With Different Frequencies

$$v_1(t) = 3.5\cos(t + 32.9^o) \quad \text{V}$$

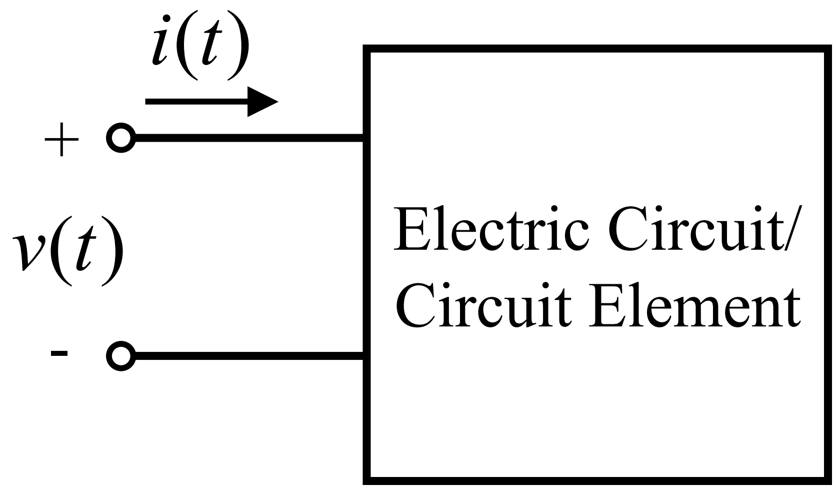
$$v_2(t) = 10.5\cos(2t + 138^o) \quad \text{V}$$

$$v(t) = v_1(t) + v_2(t) = 3.5\cos(t + 32.9^o) + 10.5\cos(2t + 138^o)$$

→ Note we can only add waveforms at different frequencies in the time domain. We **CANNOT** add their phasors. Each phasor is defined at a specific frequency and phasors with different frequencies cannot be added.

Instantaneous Power

Passive Sign Convention:



→ Power ***supplied*** to circuit/circuit element:

$$p(t) = i(t)v(t)$$

→ In the case of sinusoidal excitation in the steady state:

$$v(t) = V_m \cos(\omega t + \theta_V) \qquad i(t) = I_m \cos(\omega t + \theta_I)$$

$$p(t) = I_m V_m \cos(\omega t + \theta_V)\cos(\omega t + \theta_I)$$

→ Recall: $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha + \beta) + \cos(\alpha - \beta)\right]$

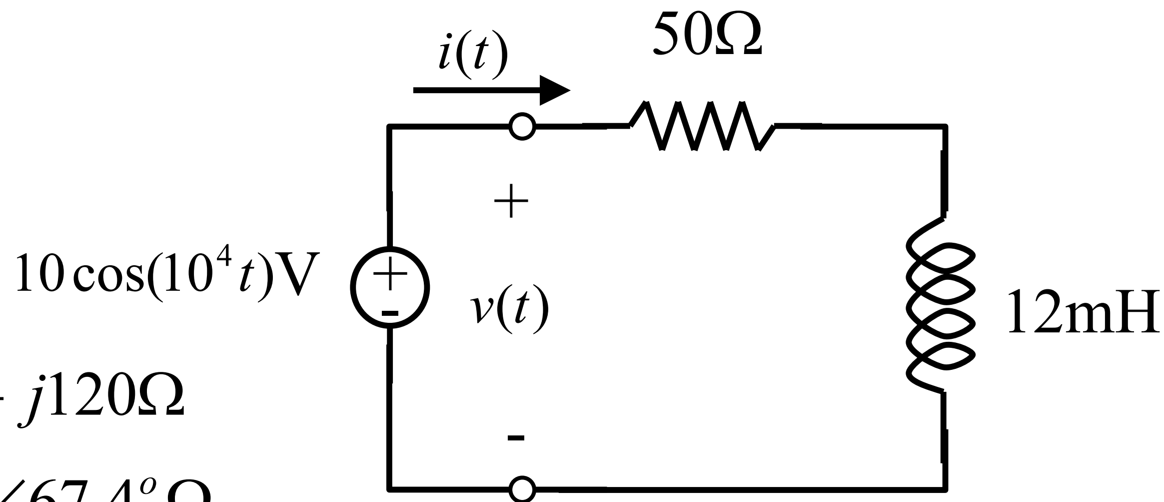
Instantaneous Power

$$p(t) = I_m V_m \cos(\omega t + \theta_V)\cos(\omega t + \theta_I)$$

$$p(t) = \underbrace{\frac{I_m V_m}{2}\cos(\theta_V - \theta_I)}_{\text{Constant}} + \underbrace{\frac{I_m V_m}{2}\cos(2\omega t + \theta_V + \theta_I)}_{\text{Periodic in time}}$$

1. For circuits with time varying inputs (e.g., sinusoidal input) the **instantaneous** power is a function of time denoted as $p(t)$.
2. For circuits with sinusoidal inputs in the **steady state**, $p(t)$ has a constant component and a periodic component.

Example 1



$$v(t) = 10\cos(10^4 t)\ \text{V}$$

$$\mathbf{V} = 10\angle 0^o\ \text{V}$$

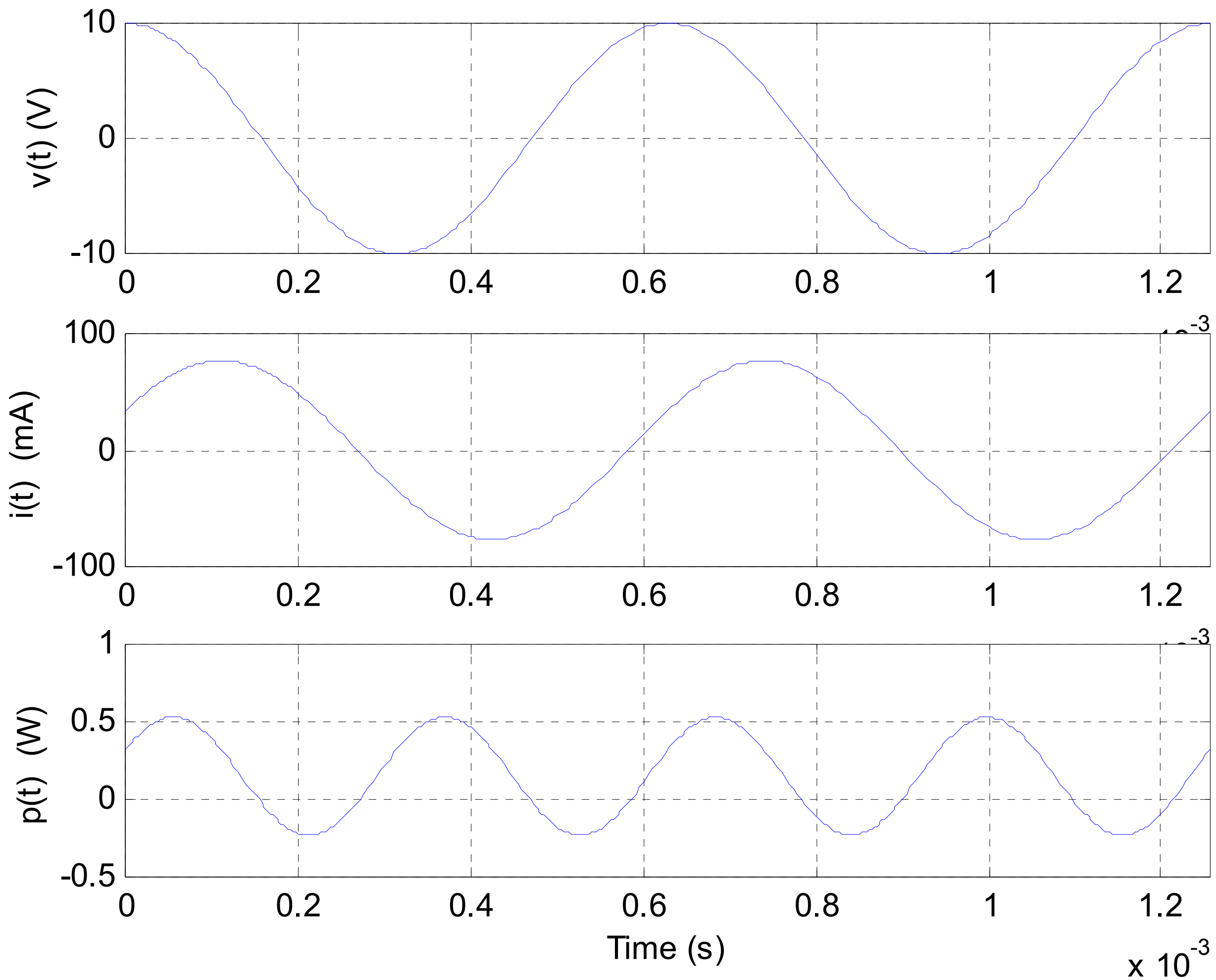
$$Z = 50\Omega + j\omega L\Omega = 50 + j120\Omega$$
$$= 130\angle 67.4^o\,\Omega$$

$$\mathbf{I} = \frac{\mathbf{V}}{Z} = \frac{10\angle 0^o}{130\angle 67.4^o} = 76.9\angle -67.4^o$$

$$i(t) = 76.9\cos(10^4 t - 67.4^o)\ \text{mA}$$

$$p(t) = \left[76.9\cos(10^4 t - 64^o)\right]\left[10\cos(10^4 t)\right]$$
$$= \frac{769}{2}\cos(67.4^o) + \frac{769}{2}\cos(2\times 10^4 t - 67.4^o)\ \ \text{mW}$$

Example 1



$$p(t) = 0.15 + 0.38\cos(2\times 10^4 t - 67.4^o) \quad \text{W}$$

Average Power

→ In the case of periodic excitation in the steady-state, the ***instantaneous power*** $p(t)$ is a periodic function of time.

→ The average value of p(t) over one period (the ***average power***) is a useful indicator of how much power is absorbed/supplied.

→ The *average power* is NOT a function of time for a periodic excitation!

→ For a general periodic function $x(t)$, such that $x(t+T)=x(t)$ the average value X is given by

$$X_{ave} = \frac{1}{T}\int_{t_o}^{t_o+T} x(\tau)\,d\tau$$

Average Power

→ In the case of Example 1:

$$p(t) = 0.15 + 0.38\cos(2\times10^4 t - 67.4^o)$$

$$P_{avg} = \frac{1}{T}\int_0^T p(\tau)d\tau = \underbrace{\frac{1}{T}\int_0^T 0.15 d\tau}_{0.15} + \underbrace{\frac{1}{T}\int_0^T 0.38\cos(2\times10^4\tau - 67.4^o)d\tau}_{\textbf{zero}}$$

$$P_{avg} = 0.15 \text{ W}$$

Average Power

→ General case

$$p(t) = \underbrace{\frac{I_m V_m}{2}\cos(\theta_V - \theta_I)}_{\text{constant}} + \frac{I_m V_m}{2}\cos(2\omega t + \theta_V + \theta_I)$$

constant

$$P_{avg} = \frac{1}{T}\int_0^T \frac{I_m V_m}{2}\cos(\theta_V - \theta_I)\,d\tau + \underbrace{\frac{1}{T}\int_0^T \frac{I_m V_m}{2}\cos(2\omega\tau + \theta_V + \theta_I)\,d\tau}_{\text{zero}}$$

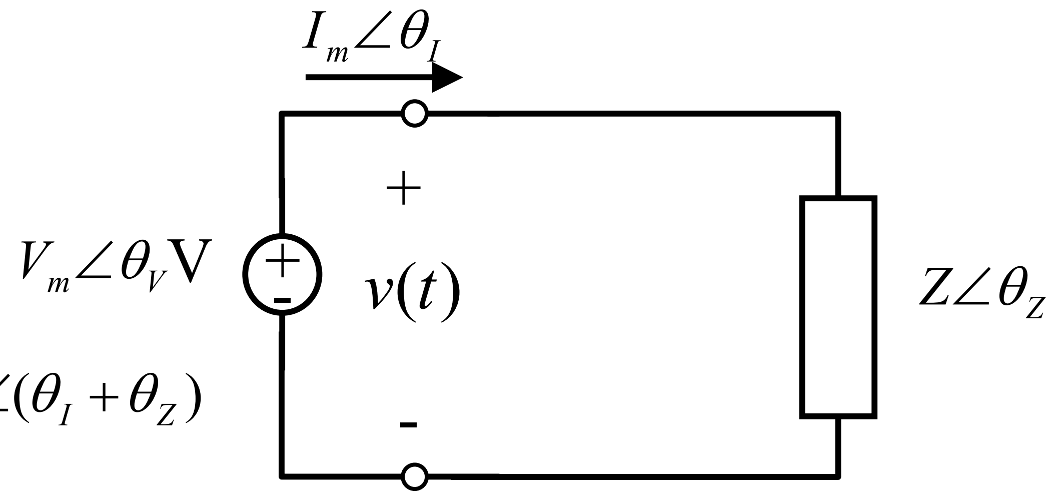
zero

$$P_{avg} = \frac{I_m V_m}{2}\cos(\theta_V - \theta_I)$$

Average Power Consumed by a General Load

$\mathbf{Z} = Z\angle\theta_Z$

$$P_{avg} = \frac{I_m V_m}{2}\cos(\theta_V - \theta_I)$$



$$V_m\angle\theta_V = (I_m\angle\theta_I)(Z\angle\theta_Z) = I_m Z\angle(\theta_I + \theta_Z)$$

$V_m = I_m Z$ $\qquad \theta_V = \theta_I + \theta_Z$

$I_m = \frac{V_m}{Z}$ $\qquad \theta_I = \theta_V - \theta_Z$

→ Substitute into power equation:

$$P_{avg} = \frac{1}{2} I_m^2 Z \cos(\theta_Z)$$

and

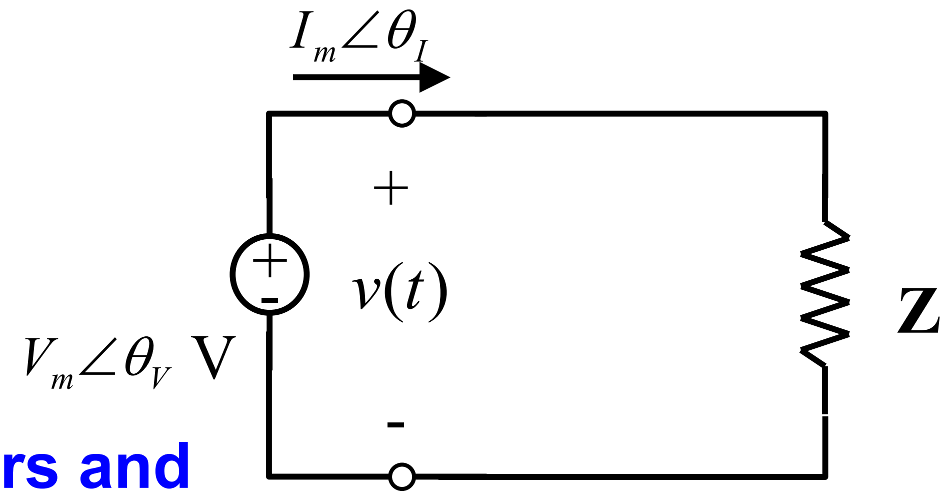
$$P_{avg} = \frac{1}{2}\frac{V_m^2}{Z}\cos(\theta_Z)$$

Average Power in a Resistor

For ac

If $\mathbf{Z} = R$

$$P_{avg} = \frac{1}{2} I_m^2 R \cos(0) = \frac{1}{2} I_m^2 R$$



Note that by using phasors and impedances we are implicitly assuming sinusoidal signals.

→ **In the case of dc**, the average power in a resistor is:

$$P_{dc} = i_{dc}^2 R$$

→ What is the ***dc current*** that produces the same average power as ac case?

$$\frac{I_m}{\sqrt{2}}$$

Root Mean Square

We define the ***Effective*** or ***Root Mean Square (RMS)*** value of a sinusoidal voltage or current waveform as the amplitude divided by the square root of 2:

$$I_{rms} = \frac{I_m}{\sqrt{2}} \qquad V_{rms} = \frac{V_m}{\sqrt{2}}$$

In this case, the average power in a general impedance is given by:

$$P_{avg} = \frac{I_m V_m}{2}\cos(\theta_V - \theta_I) = I_{rms} V_{rms} \cos(\theta_V - \theta_I)$$

$$P_{avg} = I_{rms}^2 Z \cos(\theta_Z) \qquad P_{avg} = \frac{V_{rms}^2}{Z}\cos(\theta_Z)$$

RMS Value of a Periodic Waveform

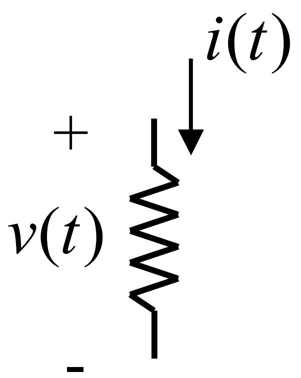
1. We looked at the RMS or effective value of sinusoidal wave forms.
2. This is a useful way of comparing average power due to general periodic waveforms.

→ Consider the instantaneous power in a resistor:

$$p(t) = i^2(t)R$$

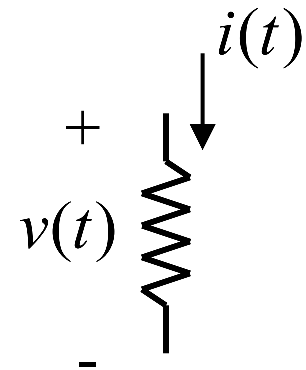
→ The average power is:

$$P_{avg} = \frac{1}{T}\int_0^T p(\tau)d\tau = \frac{1}{T}\int_0^T i^2(\tau)Rd\tau$$



RMS Value of a Periodic Waveform

$$P_{avg} = \frac{1}{T}\int_0^T i^2(\tau) R\, d\tau = R\left[\frac{1}{T}\int_0^T i^2(\tau)\, d\tau\right] = RI_{rms}^2$$



The ***effective*** value of the periodic waveform *i(t)* is the dc current that results in the **same average power**:

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(\tau)\, d\tau}$$

The above quantity is the ***root*** of the ***mean*** of the ***square*** of *i(t)* also known as the ***Root Mean Square (RMS)*** or effective value.

RMS Value of a Sine Wave

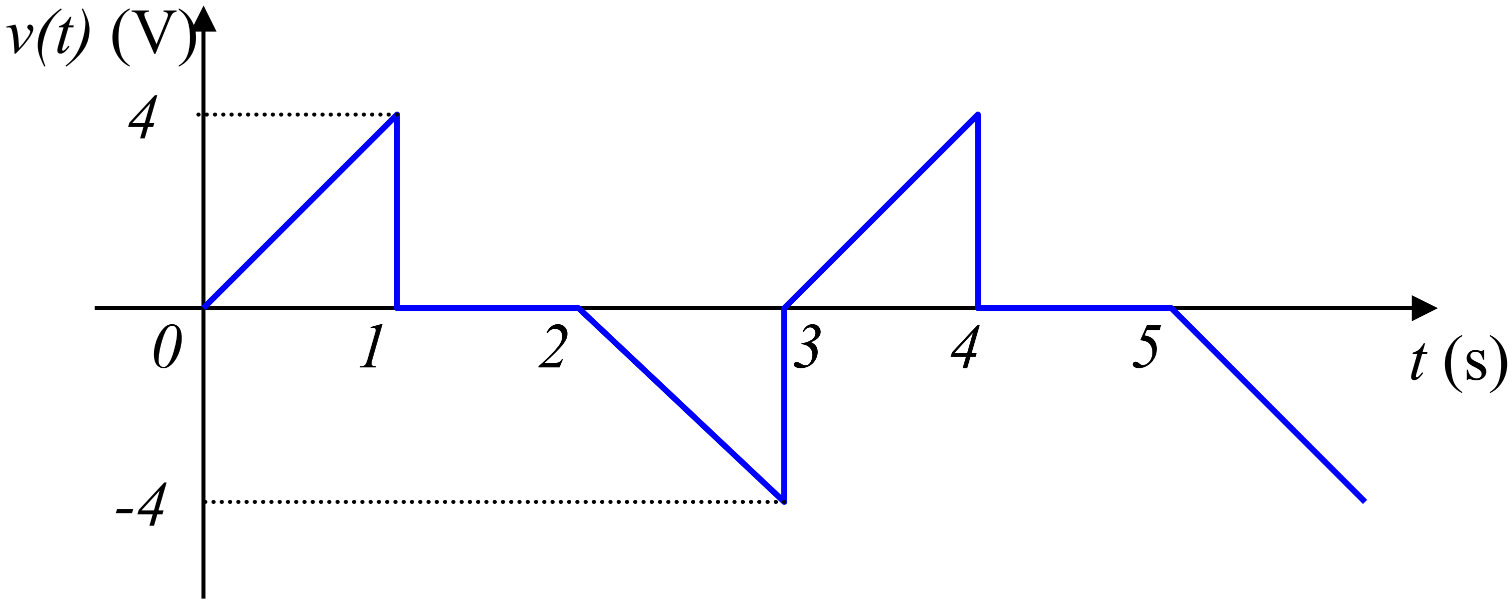
$$i(t) = I_m \cos(\omega t + \theta_I) \qquad T = \frac{2\pi}{\omega}$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(\tau)d\tau} = \sqrt{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} I_m^2 \cos^2(\omega\tau + \theta_I)d\tau}$$

Recall: $\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$

$$I_{rms} = I_m \sqrt{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left[\frac{1}{2} + \frac{1}{2}\cos(2\omega\tau + 2\theta_I)\right]d\tau} = \frac{I_m}{\sqrt{2}}$$

RMS Example



$$v(t) = \begin{cases} 4t \ \text{V} & 0 < t \leq 1 \\ 0 \ \text{V} & 1 < t \leq 2 \\ -4t+8 \ \text{V} & 3 < t \leq 3 \end{cases}$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(\tau)\,d\tau}$$

RMS Example

Since

$$v(t)=\begin{cases} 4t\ \text{V} & 0<t\le 1 \\ 0\ \text{V} & 1<t\le 2 \\ -4t+8\ \text{V} & 2<t\le 3 \end{cases}$$

$$V_{rms}=\sqrt{\frac{1}{3}\left[\int_0^1 (4\tau)^2 d\tau+\int_1^2 (0)^2 d\tau+\int_2^3 (-4\tau+8)^2 d\tau\right]}$$

$$V_{rms}=\sqrt{\frac{1}{3}\left[\left.\frac{16t^3}{3}\right|_0^1+\left.\left(64t-\frac{64t^2}{2}+\frac{16t^3}{3}\right)\right|_2^3\right]}=1.89V$$

RMS Conclusion

→ The RMS value of a periodic waveform is a **measure** to compare the *effectiveness* of sources in delivering power to a resistive load.

→ The RMS or *effective* value of a waveform is the value of the *dc* waveform that supplies the same average power to a resistive load.

→ In the case of sinusoidal waveforms, we have shown that the RMS value is given by:

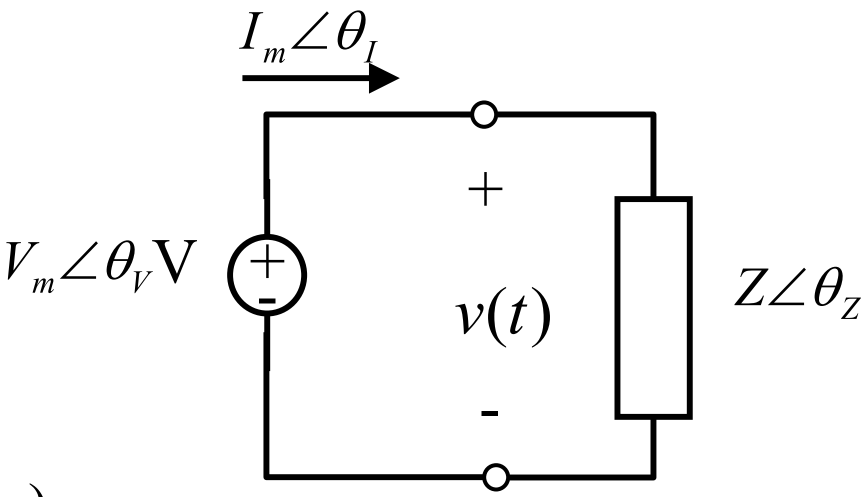
$$i(t) = I_m \cos(\omega t + \theta_I)$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

RMS Phasor

$$\mathbf{V_{rms}} = \frac{V_m}{\sqrt{2}}\angle\theta_V$$

$$\mathbf{I_{rms}} = \frac{I_m}{\sqrt{2}}\angle\theta_I$$



$$V_m\angle\theta_V = (I_m\angle\theta_I)(Z\angle\theta_Z) = I_m Z\angle(\theta_I + \theta_Z)$$

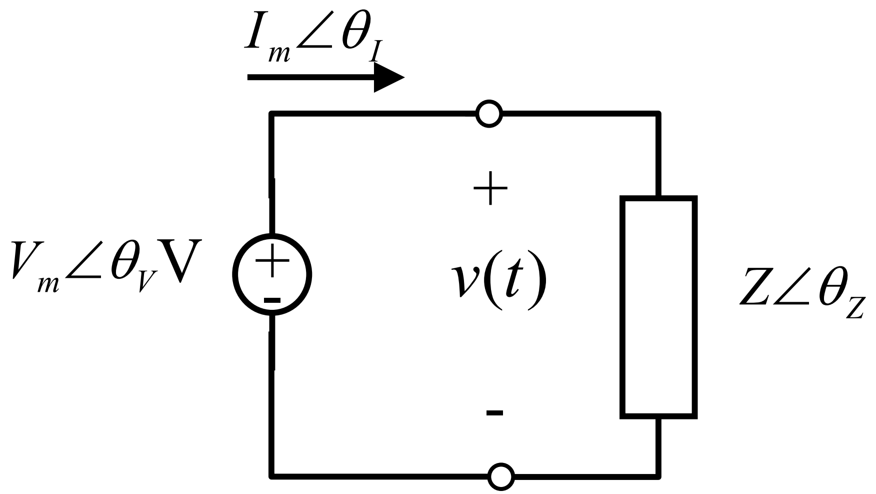
$$V_{rms}\angle\theta_V = (I_{rms}\angle\theta_I)(Z\angle\theta_Z) = I_{rms} Z\angle(\theta_I + \theta_Z)$$

→ Note: The impedance is still the same!

→ Example: Find the RMS phasor of: $v(t) = 4\cos(4t + 30^o)$

$$\mathbf{V_{rms}} = V_{rms}\angle\theta_v = \frac{4}{\sqrt{2}}\angle 30^o$$

AC Steady-State Power



$$P_{avg} = \frac{I_m V_m}{2}\cos(\theta_V - \theta_I)$$

$$P_{avg} = \frac{1}{2} I_m^2 Z\cos(\theta_Z)$$

$$P_{avg} = \frac{1}{2}\frac{V_m^2}{Z}\cos(\theta_Z)$$

$$P_{avg} = I_{rms} V_{rms}\cos(\theta_V - \theta_I)$$

$$P_{avg} = I_{rms}^2 Z\cos(\theta_Z)$$

$$P_{avg} = \frac{V_{rms}^2}{Z}\cos(\theta_Z)$$