ECSE 210: Circuit Analysis

Lecture #12:

Impedance, Admittance, & Phasor Circuits

Circuit Elements Summary

Resistor: $\mathbf{V} = R\mathbf{I}$

Inductor: $\mathbf{V} = j\omega L\mathbf{I}$

Capacitor $\mathbf{I} = j\omega C\mathbf{V}$ or $\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$

In general: $\mathbf{V} = Z\mathbf{I}$

where:

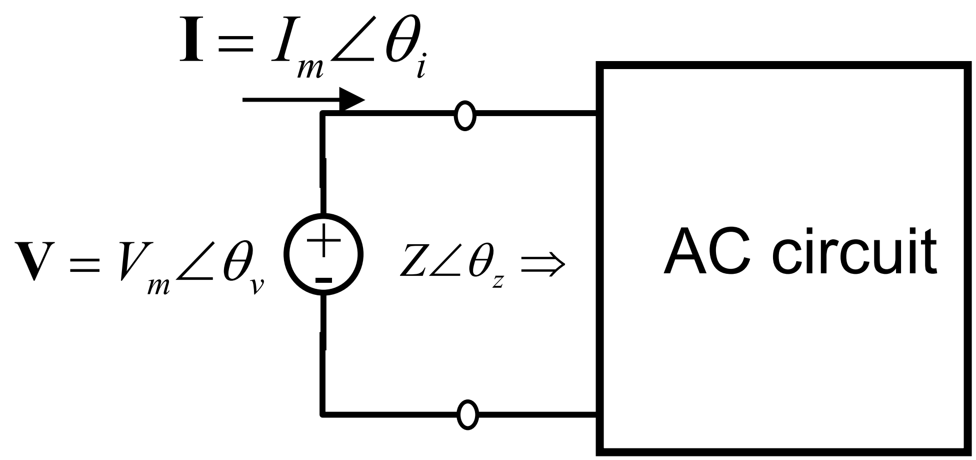
$Z = R$ → Resistor

$Z = j\omega L$ → Inductor

$Z = \dfrac{1}{j\omega C}$ → Capacitor

Z in ac circuits is analogous to R in dc circuits. It is a complex quantity but NOT a phasor!

Input Impedance



Define the two terminal *input impedance* (or *driving-point* impedance):

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = Z \angle \theta_z$$

Complex number

→ SI units for Z is Ohms (Ω)
→ **Z** in ac circuits is analogous to R in dc circuits.
→ **Z** is in general a complex quantity,
but it is **NOT a phasor!**

Impedance

The impedance is usually expressed as:

$$\mathbf{Z}(j\omega) = R(\omega) + jX(\omega)$$ **A complex quantity**

→ The real part $R(\omega)$ is the resistive component; or ***resistance***.

→ The imaginary part $X(\omega)$ is the reactive component or ***reactance***.

→ Both R and X are functions of ω (frequency).

→ $\mathbf{Z}(j\omega)$ depends on frequency.

→ In polar form: $\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta_z$

$$|\mathbf{Z}| = \sqrt{R^2 + X^2} \qquad \theta_z = \arctan\left(\frac{X}{R}\right)$$

$$R = |\mathbf{Z}|\cos(\theta_z) \qquad X = |\mathbf{Z}|\sin(\theta_z)$$

Impedance

Passive Element	Impedance
R	$\mathbf{Z} = R$
L	$\mathbf{Z} = sL = j\omega L = jX_L = \omega L \angle 90$
C	$\mathbf{Z} = \frac{1}{sC} = \frac{1}{j\omega C} = \frac{-1}{\omega C} j = jX_C = \frac{1}{\omega C} \angle -90$

$$X_L = \omega L \qquad X_C = -\frac{1}{\omega C}$$

→ Negative reactance ➔ circuit is said to be ***capacitive***.
→ Positive reactance ➔ circuit is said to be ***inductive***.

Admittance

The admittance is defined as: $\mathbf{Y}(j\omega) = \dfrac{1}{\mathbf{Z}(j\omega)}$

$\mathbf{Y}(j\omega) = G(\omega) + jB(\omega)$ → A complex quantity

→ SI units for $\mathbf{Y}$ is Siemens (S)
→ $\mathbf{Y}$ in ac circuits is analogous to G in dc circuits.
→ $\mathbf{Y}$ is in general a complex quantity,
but it is **NOT a Phasor!**

Admittance

$$\mathbf{Y}(j\omega) = G(\omega) + jB(\omega)$$ → A complex quantity

→ The real part $G(\omega)$ is the resistive component; called ***conductance***.

→ The imaginary part $B(\omega)$ is the reactive component or ***susceptance***.

→ Both G and B are functions of ω (frequency).

→ $\mathbf{Y}(j\omega)$ depends on frequency.

→ In polar form: $\mathbf{Y} = G + jB = |\mathbf{Y}|\angle\theta_Y$

$$\mathbf{Y} = \sqrt{G^2 + B^2} \qquad \theta_Y = \arctan\left(\frac{B}{G}\right)$$

$$G = |\mathbf{Y}|\cos(\theta_Y) \quad B = |\mathbf{Y}|\sin(\theta_Y)$$

Admittance

Passive Element	Admittance
R	$\mathbf{Y} = \frac{1}{R} = G$
C	$\mathbf{Y} = sC = j\omega C = jB_C = \omega C \angle 90$
L	$\mathbf{Y} = \frac{1}{sL} = \frac{1}{j\omega L} = \frac{-1}{\omega L} j = jB_L = \frac{1}{\omega L} \angle -90$

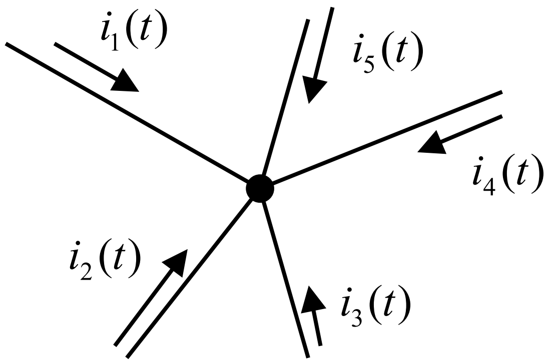
$$B_C = \omega C \qquad B_L = -\frac{1}{\omega L}$$

→ Positive susceptance ➔ circuit is said to be *capacitive*.
→ Negative susceptance ➔ circuit is said to be *inductive*.

KCL/KVL for Phasor Circuits

→ KCL in time domain:

$$i_1(t)+i_2(t)+i_3(t)+i_4(t)+i_5(t)+\text{K} \;=0$$



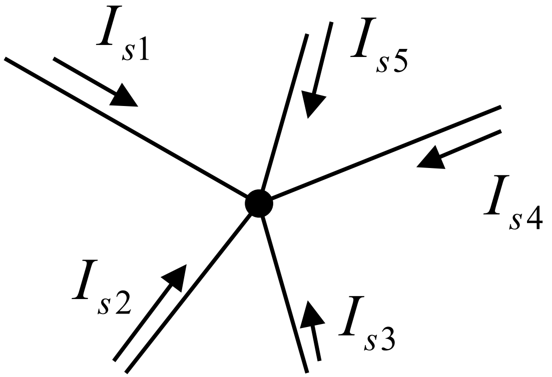
→ Complex exponential input:

$$i_1(t)+i_2(t)+i_3(t)+i_4(t)+i_5(t)+\text{K} \;=0$$

$$\mathbf{I_1}e^{j\omega t}+\mathbf{I_2}e^{j\omega t}+\mathbf{I_3}e^{j\omega t}+\mathbf{I_4}e^{j\omega t}+\mathbf{I_5}e^{j\omega t}+\text{K} \;=0$$

→ KCL for phasors

$$\mathbf{I_1}+\mathbf{I_2}+\mathbf{I_3}+\mathbf{I_4}+\mathbf{I_5}+\text{K} \;=0$$



→ We can use the same approach to prove KVL for phasor circuits.

Series and Parallel Combinations

Frequency Domain | Time Domain

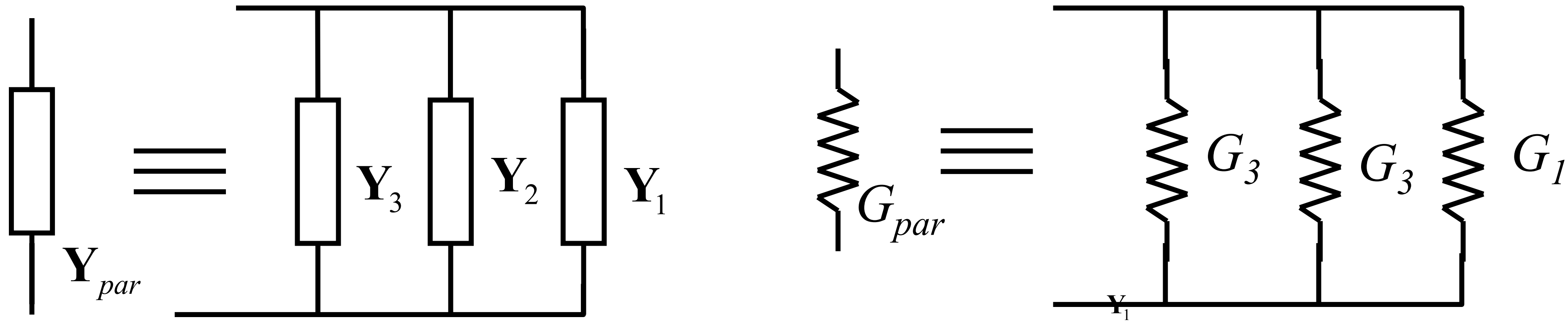
$$\left.\begin{aligned}\mathbf{V} &= \mathbf{ZI} \\ \mathbf{I} &= \mathbf{YV}\end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} v &= Ri \\ i &= Gv\end{aligned}\right.$$

→ Phasor terminal relationships are linear and in the same form as in the case of dc circuits **(except in this case we have complex numbers).**

→ KCL and KVL are valid for phasor circuits.

- ➔ We can derive network theorems that are similar to the dc case.
- ➔ Example: Element series and parallel combinations

Parallel Combinations



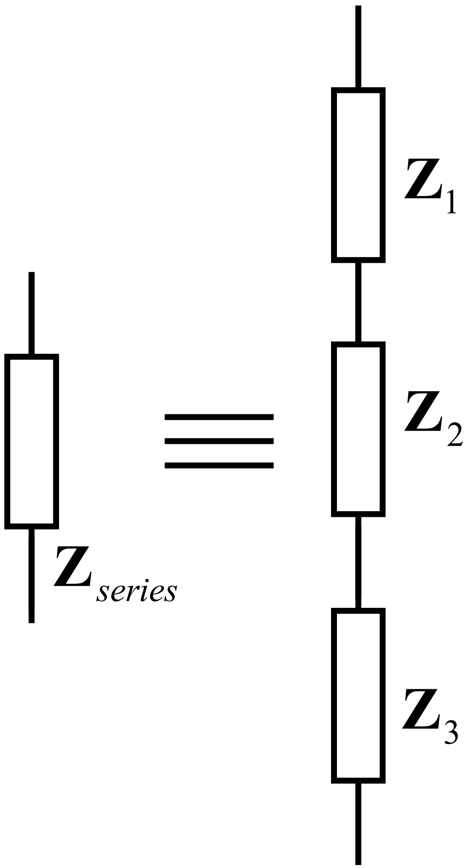
$$\mathbf{Y}_{par} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \ldots$$

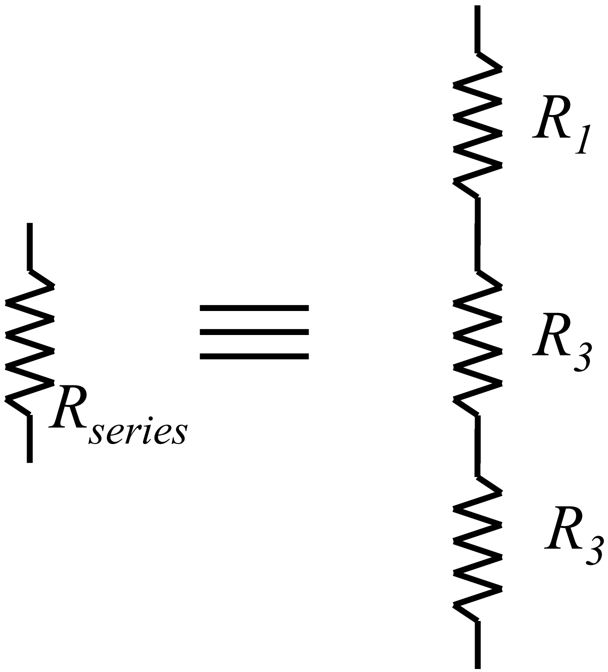
$$G_{par} = G_1 + G_2 + G_3 + \ldots$$

$$\frac{1}{\mathbf{Z}_{par}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots$$

$$\frac{1}{R_{par}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$

Series Combinations





$$\mathbf{Z}_{series} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \ldots$$

$$\frac{1}{\mathbf{Y}_{series}} = \frac{1}{\mathbf{Y}_1} + \frac{1}{\mathbf{Y}_2} + \frac{1}{\mathbf{Y}_3} + \ldots$$

$$R_{series} = R_1 + R_2 + R_3 + \ldots$$

$$\frac{1}{G_{series}} = \frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} + \ldots$$

Example 1

Calculate the equivalent impedance at 60Hz and 400Hz.

→ At 60Hz, $\omega = 2\pi f = 120\pi$

$$\mathbf{Z}_{eq} = 25\Omega + j(120\pi \times 20\times 10^{-3}) + \frac{1}{j(120\pi \times 50 \times 10^{-6})}$$

$$\mathbf{Z}_{eq} = 25\Omega + j7.54 - j53.05 = 25 - j45.51$$

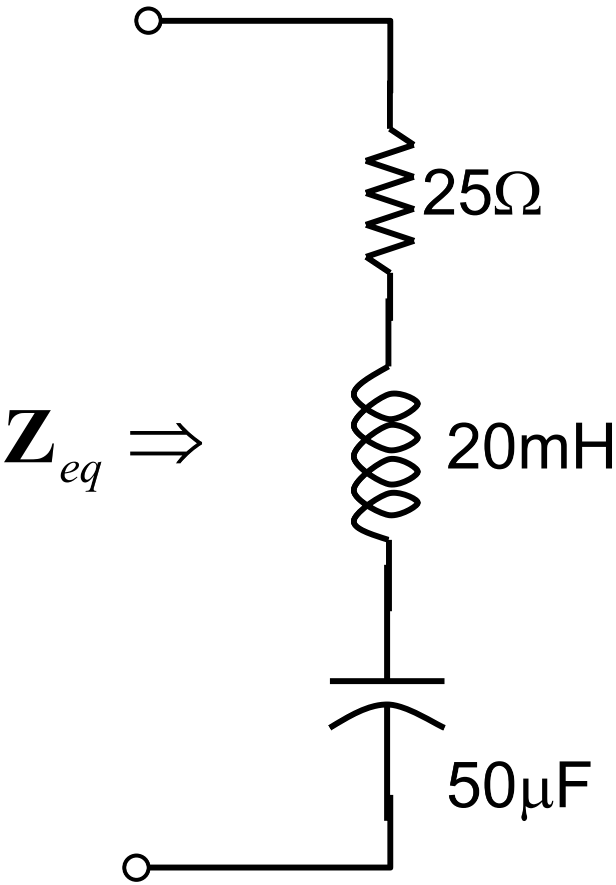
→ **Capacitive**

→ At 400Hz, $\omega = 2\pi f = 800\pi$

$$\mathbf{Z}_{eq} = 25\Omega + j(800\pi \times 20\times 10^{-3}) + \frac{1}{j(800\pi \times 50 \times 10^{-6})}$$

$$\mathbf{Z}_{eq} = 25\Omega + j50.27 - j7.96 = 25 + j42.31$$

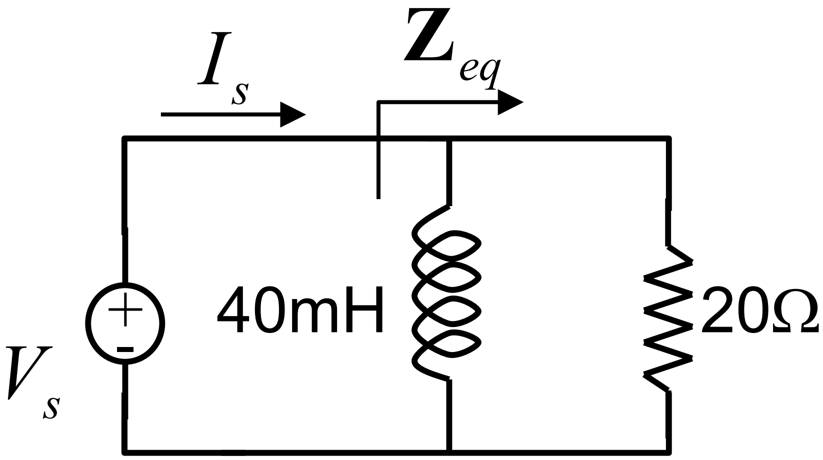
→ **Inductive**



Example 2

At 60Hz, $\mathbf{I_s} = 0.5\angle -22.98^o$

Calculate $\mathbf{V_s}$



$$\mathbf{Z}_R = 20\Omega$$

$$\mathbf{Z}_L = j\omega L = j(2\pi \times 60) \times \left(40 \times 10^{-3}\right) = j15.08\Omega$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{Z}_R \mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{(20)(j15.08)}{20 + j15.08} = 7.25 + j9.61\Omega = 12.04\angle 52.98$$

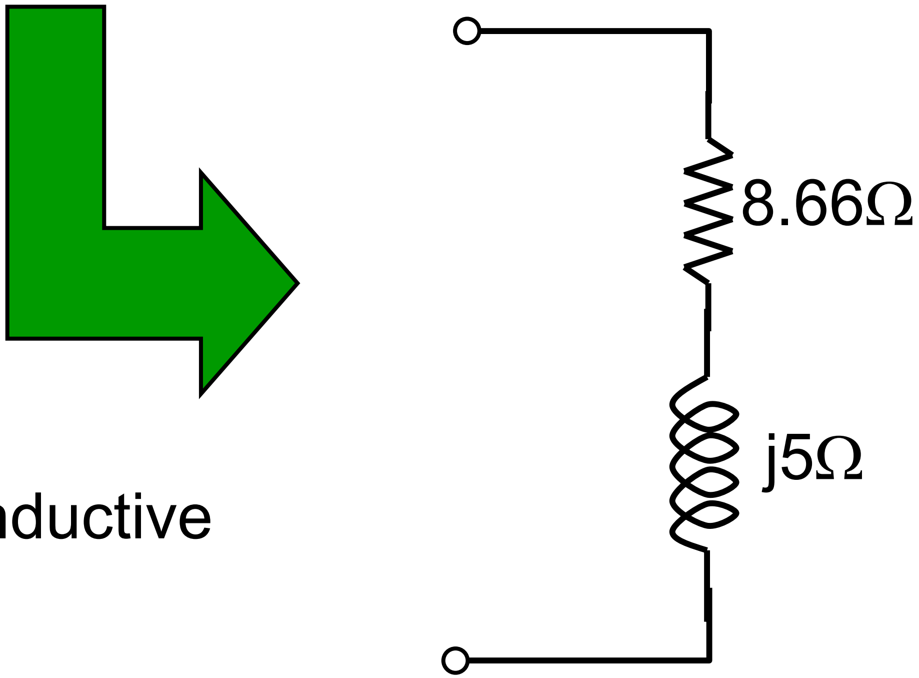
$$\mathbf{V} = \mathbf{Z}_{eq}\mathbf{I} = \left(0.5\angle -22.98^o\right)\left(12.04\angle 52.98^o\right) = 6.02\angle 30^o$$

$$v(t) = 6.02\cos(120\pi t + 30^o) \quad \text{volts}$$

Example 3

Find an equivalent circuit for the impedance: $\mathbf{Z} = 10\angle 30^o$

→ Option #1: $\mathbf{Z} = 10\angle 30^o = 8.66 + j5\Omega$



→ Positive reactance ➔ inductive

Example 3

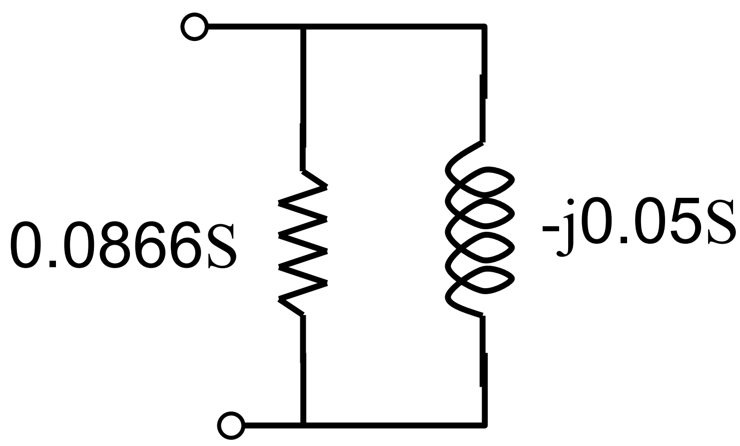
Find an equivalent circuit for the impedance: $\mathbf{Z} = 10\angle 30^o$

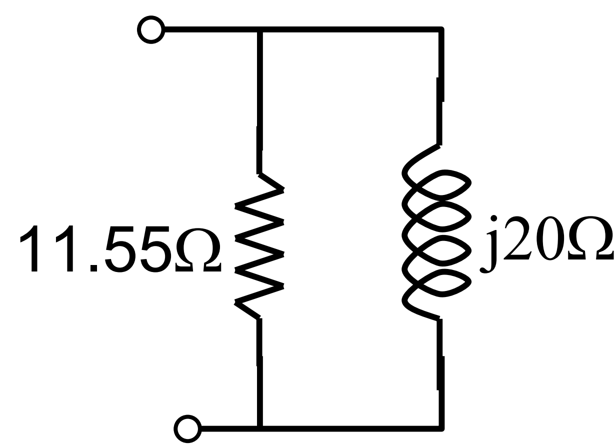
→ Option #2: $\mathbf{Z} = 10\angle 30^o = 8.66 + j5\Omega$

$$\mathbf{Y} = \frac{1}{10\angle 30^o} = 0.1\angle -30^o = 0.0866 - j0.05$$

→ Negative susceptance ➔ inductive

Y circuit





Example 3

Find an equivalent circuit for the impedance: $\mathbf{Z} = 10\angle 30^o$

→ Option #2: $\mathbf{Z} = 10\angle 30^o = 8.66 + j5\Omega$

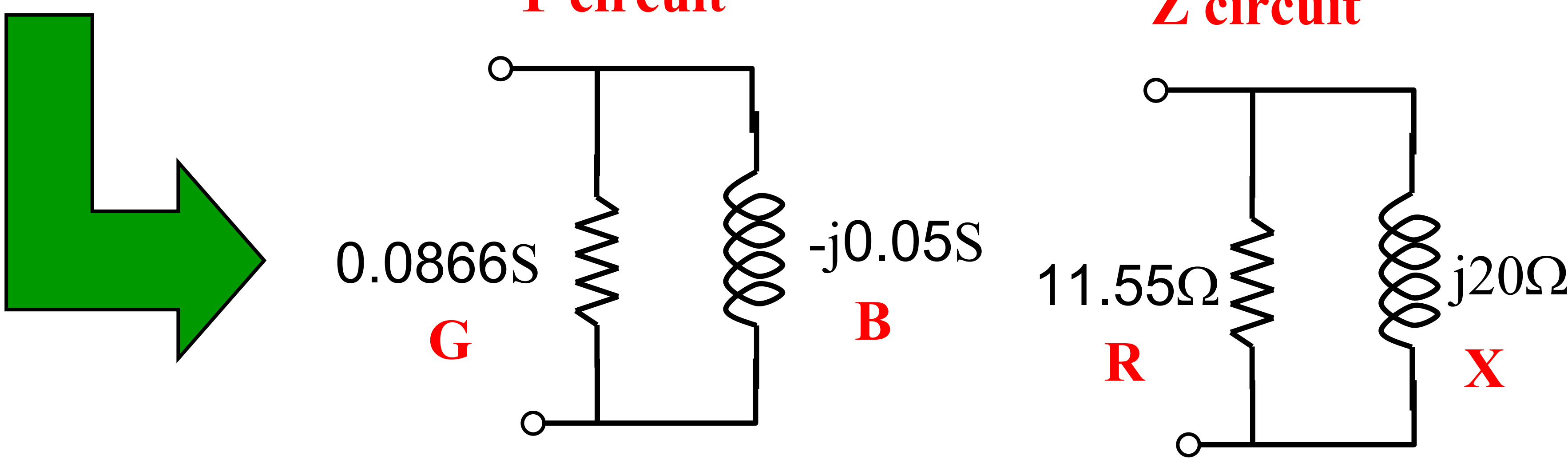
$$\mathbf{Y} = \frac{1}{10\angle 30^o} = 0.1\angle -30^o = 0.0866 - j0.05$$

equivalent circuits

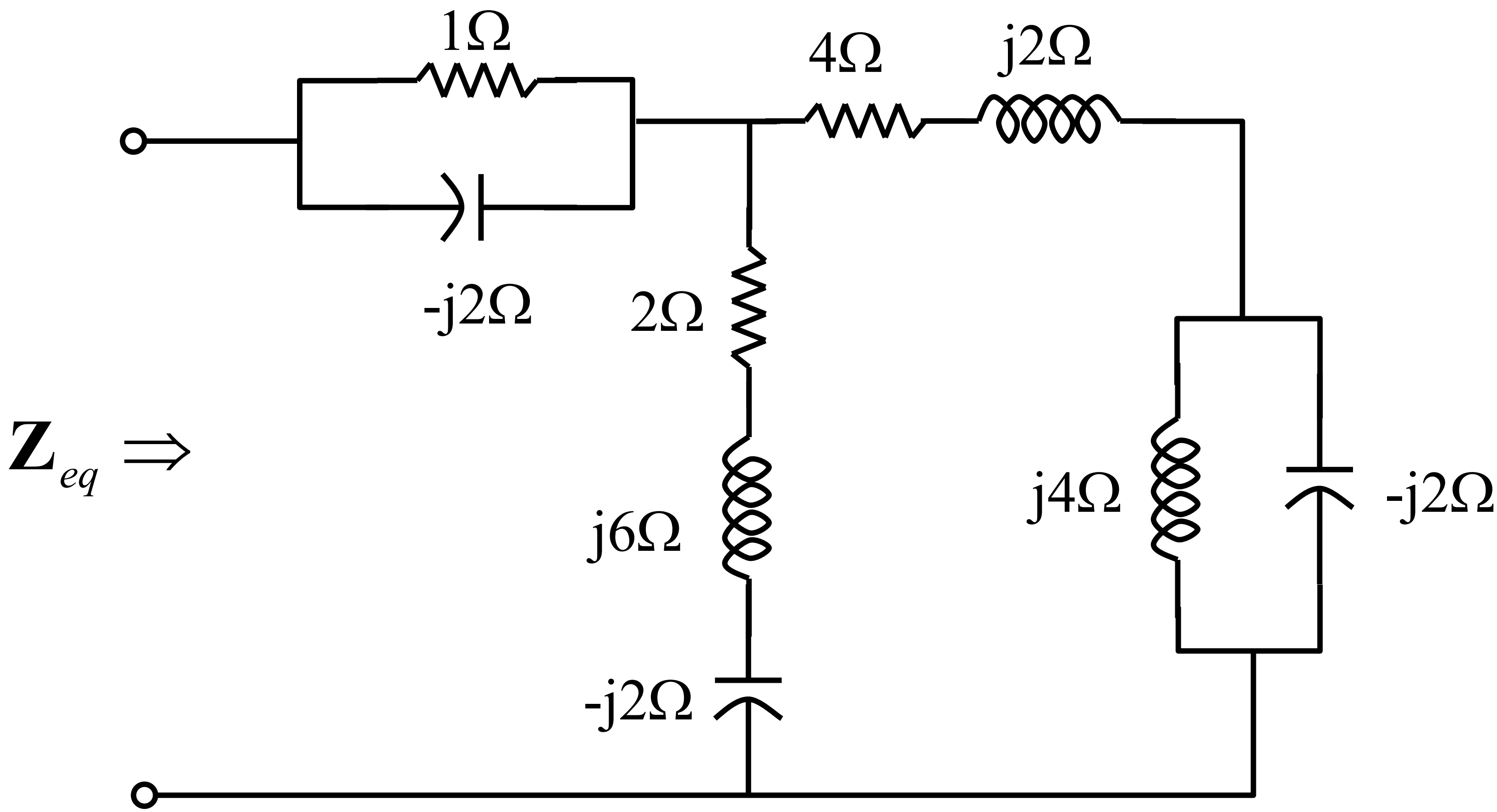
Negative susceptance → inductive

Positive reactance → inductive

Y circuit **Z circuit**



Example 4



$$\mathbf{Z}_{eq} = 1\Omega \,||\, (-2j\Omega) + \left[4\Omega + j2\Omega + j4\Omega \,||\, (-j2\Omega)\right] || \, (2\Omega + j6\Omega - j2\Omega)$$

$$\mathbf{Z}_{eq} = 3.8 + j0.6\Omega$$

AC Steady State Analysis

Basis: KCL, KVL and **V**=**ZI** can be used to solve linear, ***ac steady-state*** circuit problems.

→ Consider the network theorems (and other linear circuit analysis techniques) developed for resistive dc circuits for *ac steady-state* analysis of RLC circuits.

→ It may be confirmed that each time-domain method is valid for frequency domain analysis. (Derive equivalent frequency domain results based on the phasor form of KCL, KVL and **V**=**IZ.**)

→ We will consider a selection of illustrative *ac steady-state* (phasor) circuit analysis examples.