ECSE 210: Circuit Analysis

Lecture #11:

Complex Sources, Phasors

Complex Exponential

Maclaurin's expansion of the exponential function is:

$$e^x = 1 + x + \left(\frac{x^2}{2!}\right) + \ldots + \left(\frac{x^n}{n!}\right) + \ldots$$

Substitute the complex number $j\theta$ for x:

$$e^{j\theta} = 1 + j\theta + \left(\frac{(j\theta)^2}{2!}\right) + \left(\frac{(j\theta)^3}{3!}\right) + \ldots + \left(\frac{(j\theta)^n}{n!}\right) + \ldots$$

$$e^{j\theta} = \underbrace{\left[1 - \left(\frac{\theta^2}{2!}\right) + \left(\frac{\theta^4}{4!}\right) - \left(\frac{\theta^6}{6!}\right) + \ldots\right]}_{\cos(\theta)} + j\underbrace{\left[\theta - \left(\frac{\theta^3}{3!}\right) + \left(\frac{\theta^5}{5!}\right) - \left(\frac{\theta^7}{7!}\right) + \ldots\right]}_{\sin(\theta)}$$

Complex Exponential

→ Euler's theorem: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

→ Similarly: $e^{-j\theta} = \cos(\theta) - j\sin(\theta)$

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta) = 2\,\mathrm{Re}[e^{j\theta}]$$

$$e^{j\theta} - e^{-j\theta} = 2\sin(\theta) = 2\,\mathrm{Im}[e^{j\theta}]$$

→ We can represent a sinusoidal time varying function as the real part of a complex exponential:

$$x(t) = \mathrm{Re}[X_m e^{j(\omega t+\theta)}] = X_m \cos(\omega t + \theta)$$

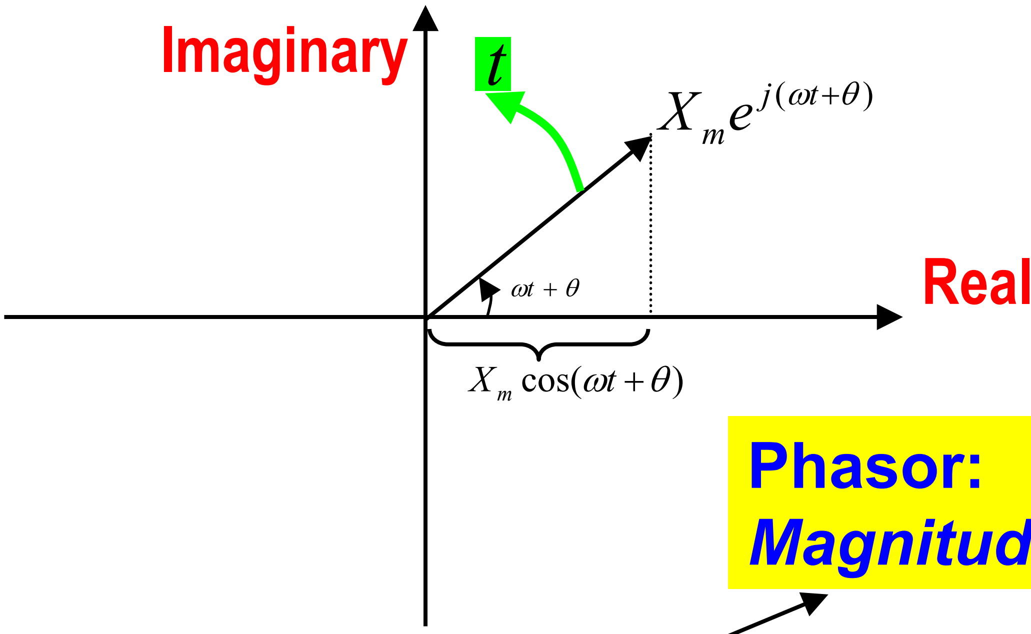
→ Note that: $X_m e^{j(\omega t+\theta)} = X_m e^{j\theta} e^{j\omega t}$

Complex Exponential

What is the difference between the following two equations?

$$X(t) = X_m e^{j(\omega t + \theta)}$$ ➡ Complex exponential function

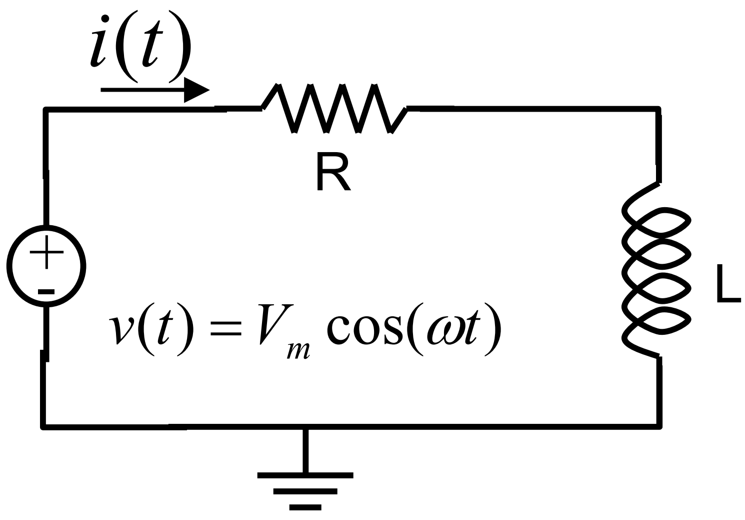
$$x(t) = \mathrm{Re}[X_m e^{j(\omega t + \theta)}] = X_m \cos(\omega t + \theta)$$



Phasor:
Magnitude & Phase

→ Note that: $X_m e^{j(\omega t + \theta)} = X_m e^{j\theta} e^{j\omega t}$

Phasor Example



→ Instead of solving this circuit, solve the same circuit but with a **complex exponential input**:

$$v(t) = \mathbf{V}\, e^{j\omega t}$$

Is this a physically realizable function?

KVL:

$$iR + L\frac{di}{dt} = \mathbf{V} e^{j\omega t}$$

Exponential forcing function

$$\mathbf{V} = V_m \angle 0$$

Phasor magnitude & phase

→ Steady state response has the form: $i(t) = \mathbf{I}\, e^{j\omega t}$

Phasor Example

→ Steady state response has the form: $i(t) = \mathbf{I}\, e^{j\omega t}$

Substitute into d.e. ➡ $R\mathbf{I}e^{j\omega t} + j\omega L\mathbf{I}\, e^{j\omega t} = \mathbf{V}\, e^{j\omega t}$

→ Divide by $e^{j\omega t}$

$R\mathbf{I} + j\omega L\mathbf{I} = \mathbf{V}$ ➡ Time-varying exponential has disappeared but that is OK. We know that for linear circuits all signals have the same form (that of the input).

$$\mathbf{I} = \frac{\mathbf{V}}{R + j\omega L} = I_m\angle\phi = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}\angle - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Phasor Example

$$\mathbf{I} = \frac{\mathbf{V}}{R + j\omega L} = I_m \angle \phi = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \angle - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Frequency domain

$$i(t) = \mathbf{I}\, e^{j\omega t} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j \tan^{-1}\left(\frac{\omega L}{R}\right)} \quad e^{j\omega t}$$

Time domain

Is this a physically realizable function?

→ Try the same circuit, but now with the input $v(t) = \mathbf{V}\, e^{-j\omega t}$

Phasor Example

If $v(t) = \mathbf{V}\, e^{j\omega t}$

$$i(t) = \mathbf{I}\, e^{j\omega t} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j \tan^{-1}(\frac{\omega L}{R})} \quad e^{j\omega t}$$

If $v(t) = \mathbf{V}\, e^{-j\omega t}$

$$i(t) = \mathbf{I}^{*} e^{-j\omega t} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j \tan^{-1}(\frac{\omega L}{R})} \quad e^{-j\omega t}$$

Phasor Example

→ **Use superposition to find the total response:**

Input: **1/2(☆+🕐)** $= \frac{1}{2}\left(V_m e^{j\omega t} + V_m e^{-j\omega t}\right) = V_m \cos(\omega t)$

Output:

1/2(🕑+🕒) $= \frac{1}{2}\frac{V_m}{\sqrt{R^2+\omega^2 L^2}}\left(e^{j\left(\omega t - \tan^{-1}(\frac{\omega L}{R})\right)} + e^{-j\left(\omega t - \tan^{-1}(\frac{\omega L}{R})\right)}\right)$

$$= \frac{V_m}{\sqrt{R^2+\omega^2 L^2}}\cos\left(\omega t - \tan^{-1}(\frac{\omega L}{R})\right)$$

Same as before.

Phasor Example

→ Therefore:

If input: $$v(t) = V_m \cos(\omega t)$$ **SINUSOID**

Then output: $$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}(\frac{\omega L}{R}) \right)$$

⇨ Note that:

$$v(t) = \mathrm{Re}[\mathbf{V}\, e^{j\omega t}]$$

$$i(t) = \mathrm{Re}[\mathbf{I}\, e^{j\omega t}]$$

Summary

Time domain input: $v(t) = V_m \cos(\omega t + \theta)$

Find complex exponential forcing function *v(t)*
such that: $\mathrm{Re}[\mathbf{V}\, e^{j\omega t}] = v(t)$

Complex forcing function: $v(t) = V_m e^{j(\omega t + \theta)} = \left(V_m \angle \theta\right) e^{j\omega t} = \mathbf{V}\, e^{j\omega t}$

Find the response *i(t)* to *v(t)*

Time domain response due to *v(t)* is:

$$i(t) = \mathrm{Re}[\mathbf{I}\, e^{j\omega t}]$$

Summary

Find the phasor of a time domain signal:

$$v(t) = V_m \cos(\omega t + \theta)$$

- Complex exponential function:

$$v(t) = V_m e^{j(\omega t + \theta)} = \left[V_m e^{j\theta}\right] e^{j\omega t} = \mathbf{V} e^{j\omega t}$$

$\mathbf{V}$ is the phasor of $v(t) = V_m \cos(\omega t + \theta)$

$$\mathbf{V} = V_m \angle \theta$$ **PHASOR**

$$v(t) = \text{Re}[\mathbf{V}\, e^{j\omega t}]$$

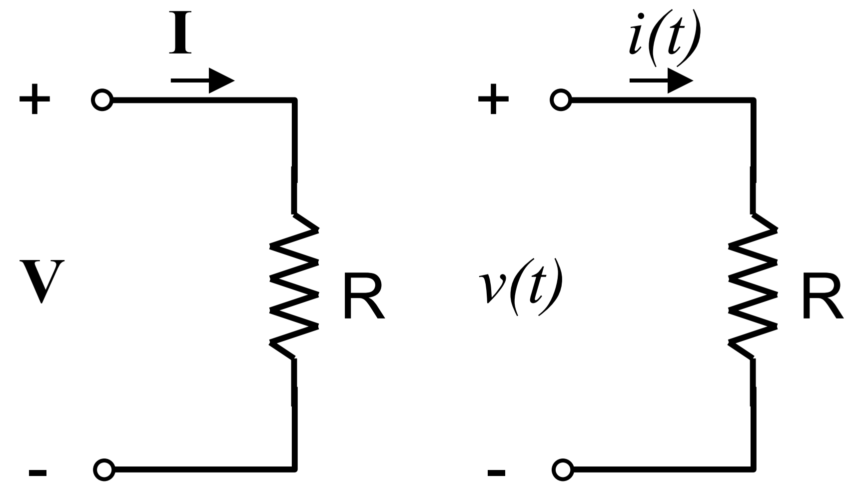
→ Note: $\mathbf{V}$ does not include the frequency term $e^{j\omega t}$.
The frequency of the steady state response is the same as that of the input in a linear circuit.

Key Points

1. Solving systems using phasors is called ***phasor analysis***, or ***frequency domain analysis***.
2. Differential equations driven by sinusoidal forcing functions in the ***time domain*** are converted to ***complex algebraic*** equations in the ***frequency domain***.
3. v(t) represents a voltage in the time domain. The phasor **V** represents the voltage in the frequency domain.
1. The phasor **V** contains only the magnitude and phase information of *v(t)*. The frequency ω is implicit to the representation.
2. It would be nice to be able to write the circuit equations *directly* in the phasor/frequency domain.

Circuit Elements in the Frequency Domain

The resistor R



→ Time domain:

$$v(t) = Ri(t)$$

→ Complex exponential forcing function:

$$v(t) = V_m e^{j(\omega t + \theta_v)} = \left[V_m \angle \theta_v \right] e^{j\omega t} = \mathbf{V}\, e^{j\omega t}$$

$$i(t) = I_m e^{j(\omega t + \theta_i)} = \left[I_m \angle \theta_i \right] e^{j\omega t} = \mathbf{I}\, e^{j\omega t}$$

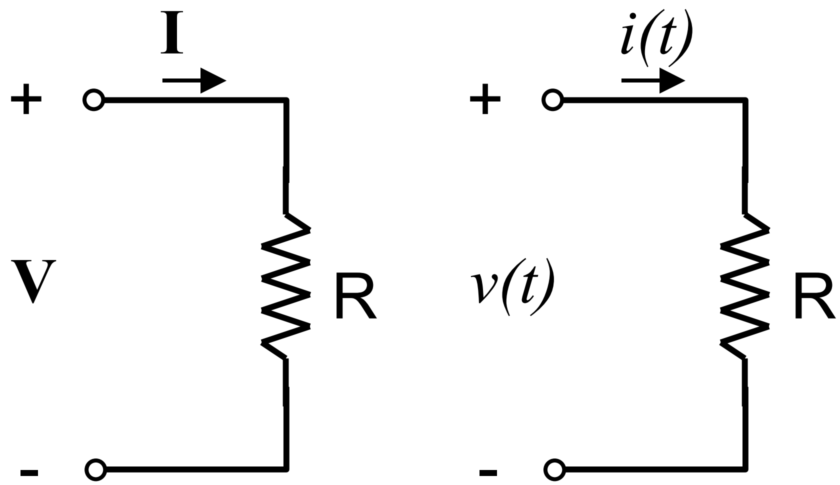
$$v(t) = Ri(t) \longrightarrow \mathbf{V}\, e^{j\omega t} = R\mathbf{I}\, e^{j\omega t}$$

→ Phasors: $\mathbf{V} = R\mathbf{I}$

Circuit Elements in the Frequency Domain

The resistor R

→ Phasors: $\mathbf{V} = R\mathbf{I}$



This terminal relation for the phasor voltage and current in a resistor implies that **the voltage and current are in phase**:

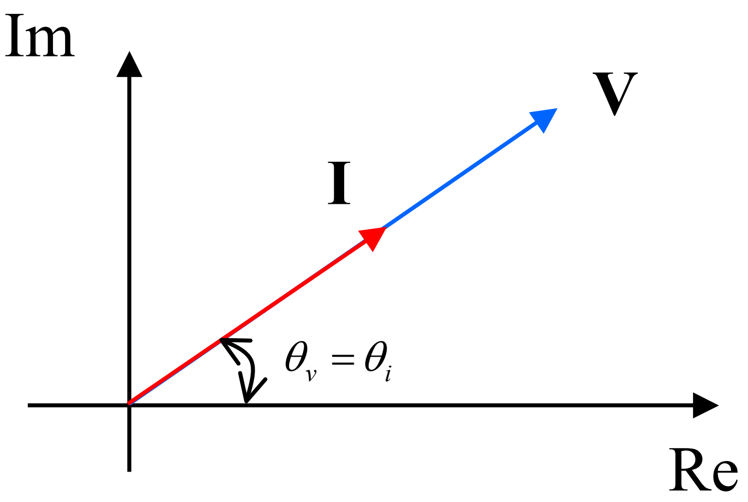
$$\mathbf{V} = R\mathbf{I}$$

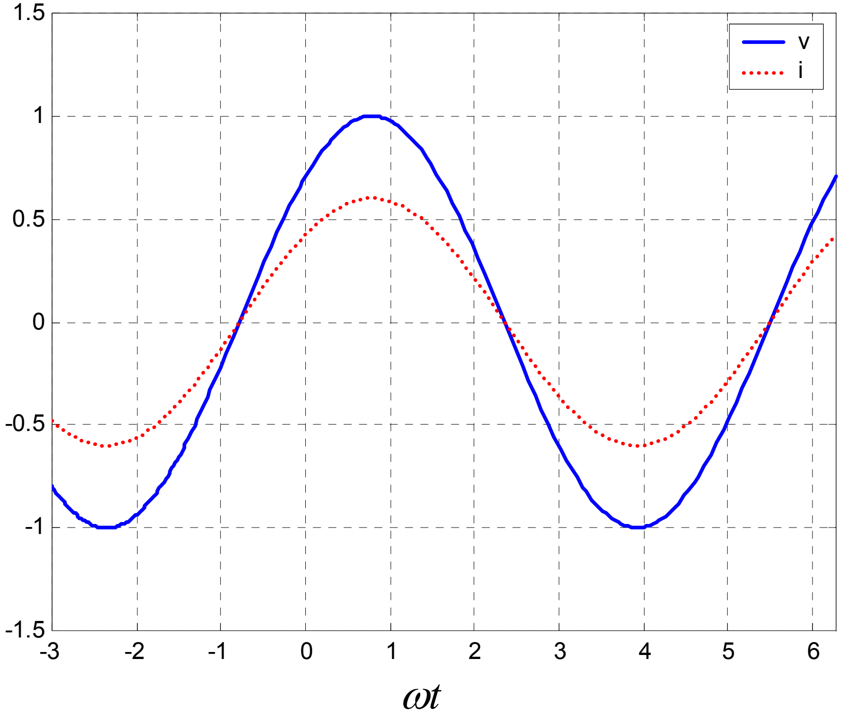
$$V_m \angle \theta_v = R I_m \angle \theta_i \longrightarrow \theta_v = \theta_i$$

Circuit Elements in the Frequency Domain

Phasor diagram:

→ Plot complex numbers as vectors

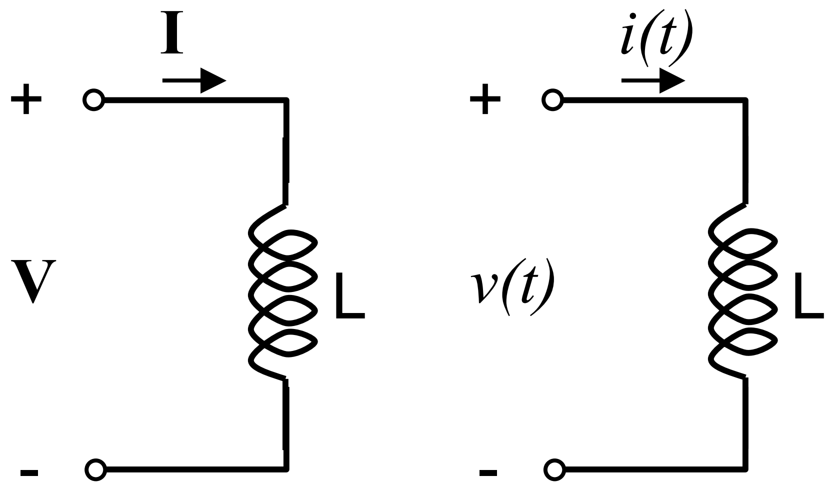




→ Voltage and current are in phase

Circuit Elements in the Frequency Domain

The inductor L



→ Time domain: $v(t) = L\dfrac{di}{dt}$

→ Complex exponential forcing function:

$$v(t) = V_m e^{j(\omega t+\theta_v)} = \left[V_m\angle\theta_v\right]e^{j\omega t} = \mathbf{V}\ e^{j\omega t}$$

$$i(t) = I_m e^{j(\omega t+\theta_i)} = \left[I_m\angle\theta_i\right]e^{j\omega t} = \mathbf{I}\ e^{j\omega t}$$

$$v(t) = L\frac{di(t)}{dt} \longrightarrow \mathbf{V}\ e^{j\omega t} = j\omega L\mathbf{I}\ e^{j\omega t}$$

→ Phasors: $\mathbf{V} = j\omega L\mathbf{I}$

Circuit Elements in the Frequency Domain

The inductor L

→ Phasors: $\mathbf{V} = j\omega L\mathbf{I}$

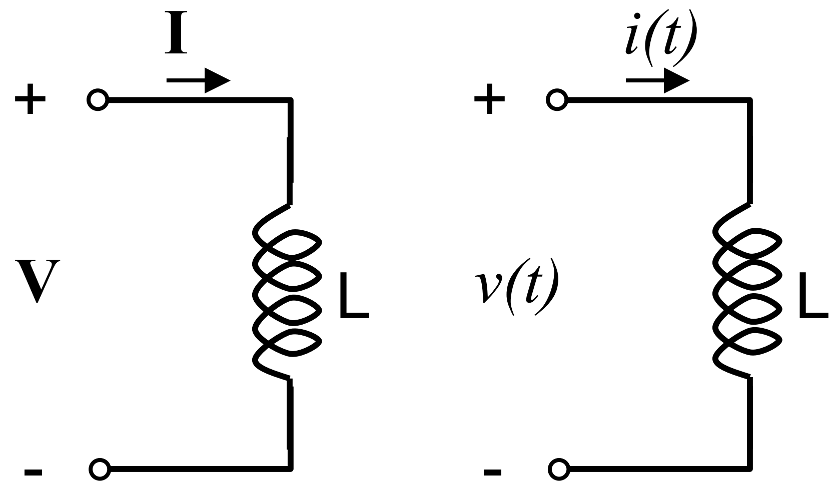
$$V_m\angle\theta_v = j\omega L I_m\angle\theta_i$$

$$j = 1\angle 90^o$$

$$V_m\angle\theta_v = \omega L I_m\angle\theta_i + 90^0$$

$$\theta_v = \theta_i + 90^0 \qquad \theta_v - \theta_i = +90^0$$

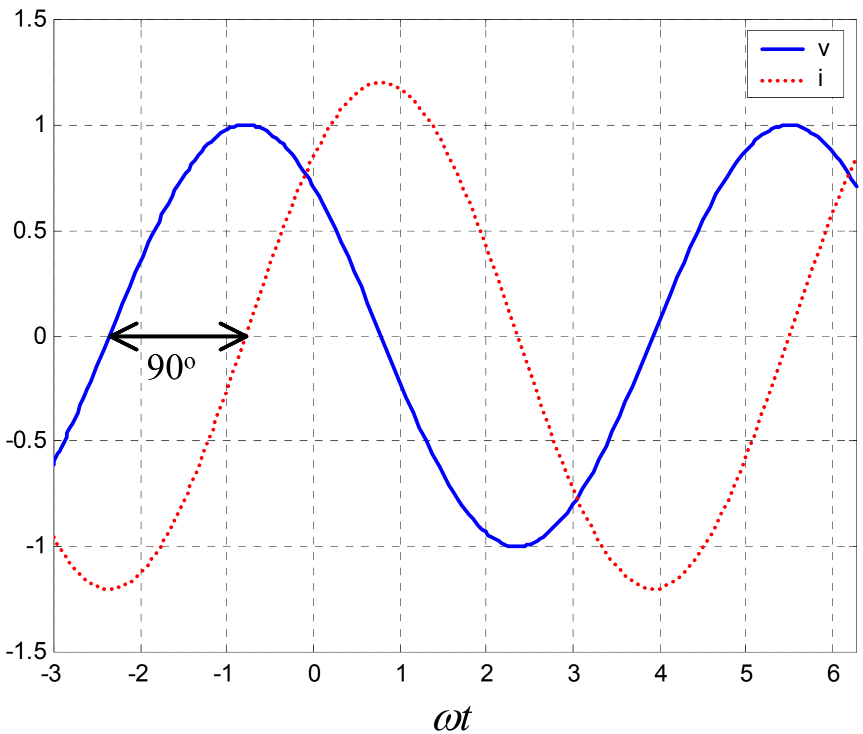
→ *v(t)* leads *i(t)* by 90°

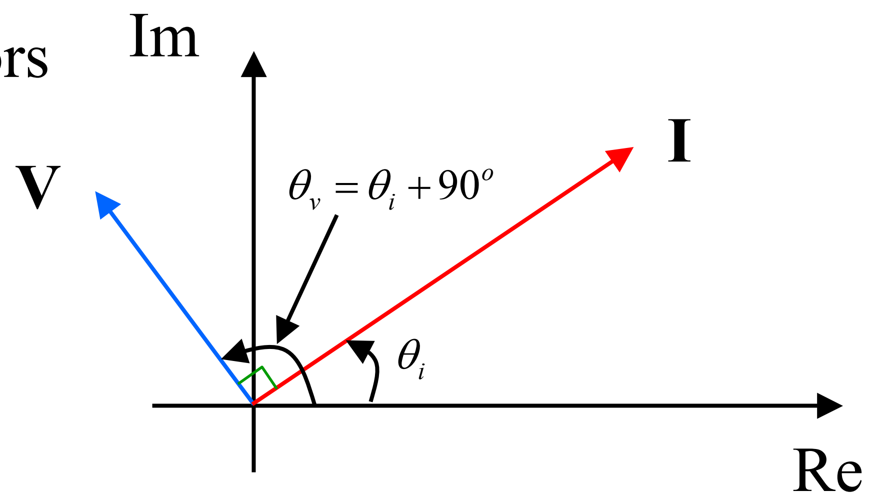


Circuit Elements in the Frequency Domain

Phasor diagram:

→ Plot complex numbers as vectors



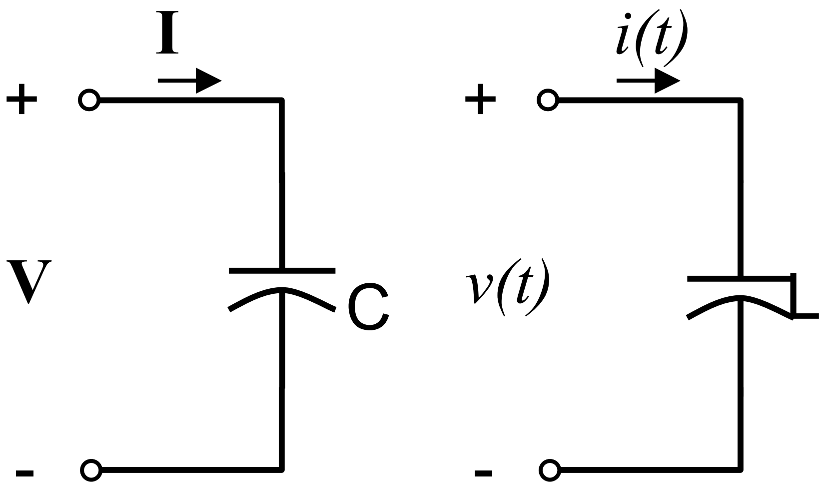


→ Voltage leads current by 90 degrees.

Circuit Elements in the Frequency Domain

The capacitor C

→ Time domain: $i(t) = C\dfrac{dv}{dt}$



→ Complex exponential forcing function:

$$v(t) = V_m e^{j(\omega t+\theta_v)} = \left[V_m\angle\theta_v\right]e^{j\omega t} = \mathbf{V}\ e^{j\omega t}$$

$$i(t) = I_m e^{j(\omega t+\theta_i)} = \left[I_m\angle\theta_i\right]e^{j\omega t} = \mathbf{I}\ e^{j\omega t}$$

$$i(t) = C\frac{dv(t)}{dt} \longrightarrow \mathbf{I}\ e^{j\omega t} = j\omega C\mathbf{V}\ e^{j\omega t}$$

→ Phasors: $\mathbf{I} = j\omega C\mathbf{V}$

Circuit Elements in the Frequency Domain

The capacitor C

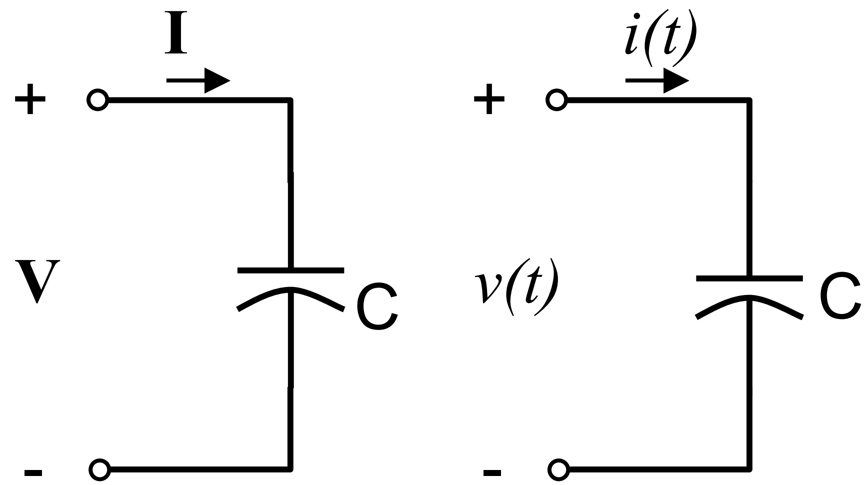
→ Phasors: $\mathbf{I} = j\omega C\mathbf{V}$

$$I_m \angle \theta_i = j\omega C V_m \angle \theta_v$$

$$I_m \angle \theta_i = \omega C V_m \angle \theta_v + 90^o$$

$$\theta_i = \theta_v + 90^o \qquad \theta_v - \theta_i = -90^o$$

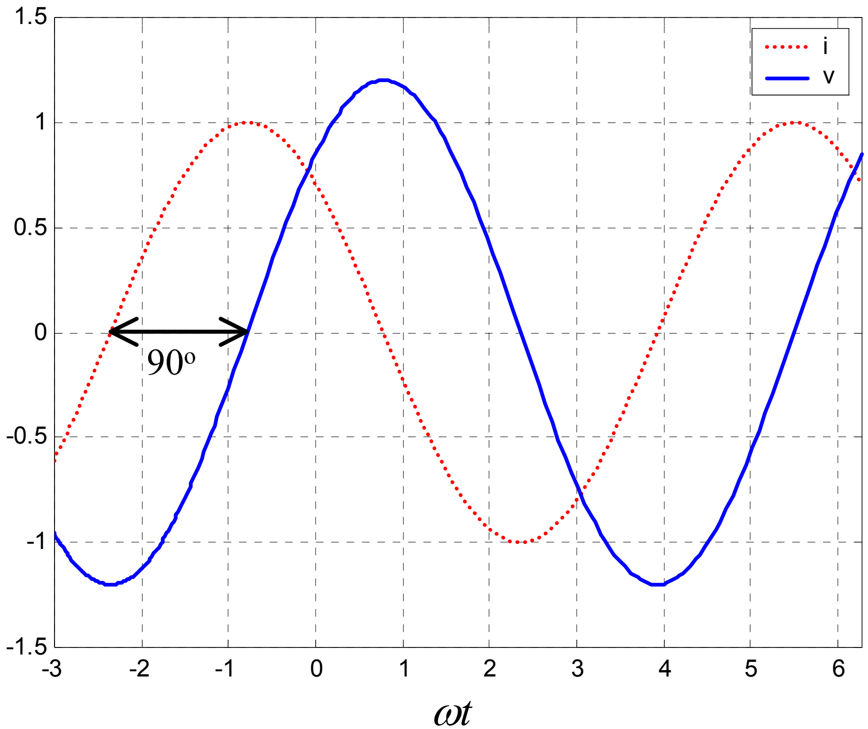
→ *v(t)* lags *i(t)* by 90°

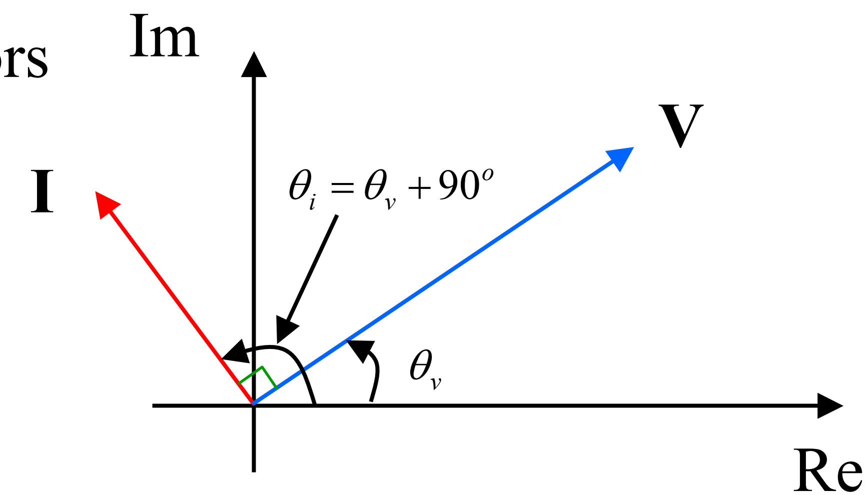


Circuit Elements in the Frequency Domain

Phasor diagram:

→ Plot complex numbers as vectors





→ Voltage lags current by 90 degrees.

Circuit Elements: Summary

Resistor: $\mathbf{V} = R\mathbf{I}$

Inductor: $\mathbf{V} = j\omega L\mathbf{I}$

Capacitor $\mathbf{I} = j\omega C\mathbf{V}$ or $\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$

In general: $\mathbf{V} = Z\mathbf{I}$

where:

$Z = R$ → Resistor

$Z = j\omega L$ → Inductor

$Z = \frac{1}{j\omega C}$ → Capacitor

Z in ac circuits is analogous to R in dc circuits. It is a **complex** quantity but NOT a phasor!