ECSE 210: Circuit Analysis

Lecture #10:

→ Second Order Circuits Conclusion

→ Sinusoids

→ Circuits with Sinusoidal Forcing Functions

RLC Circuits: Final Remarks

1. RLC circuit analysis leads to second order ODEs.
2. RLC circuit response is determined by the roots of the characteristic equation of the governing ODE:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

where: α is the exponential damping coefficient and ω_o is the undamped resonant frequency

3. *Real* and *unequal* roots ($\alpha > \omega_0$) → overdamped;
   Complex &*unequal* roots ($\alpha < \omega_o$) → underdamped;
   Real and *equal* roots ($\alpha = \omega_o$) → critically damped.
4. The general response/solution of even very simple RLC circuits can be very difficult to determine in the time domain.

→ Use *transform* methods which simplify the analysis.

Sinusoids

The sinusoidal forcing function is a very important electric circuit excitation:

1. It is the dominant waveform in the electric power industry.

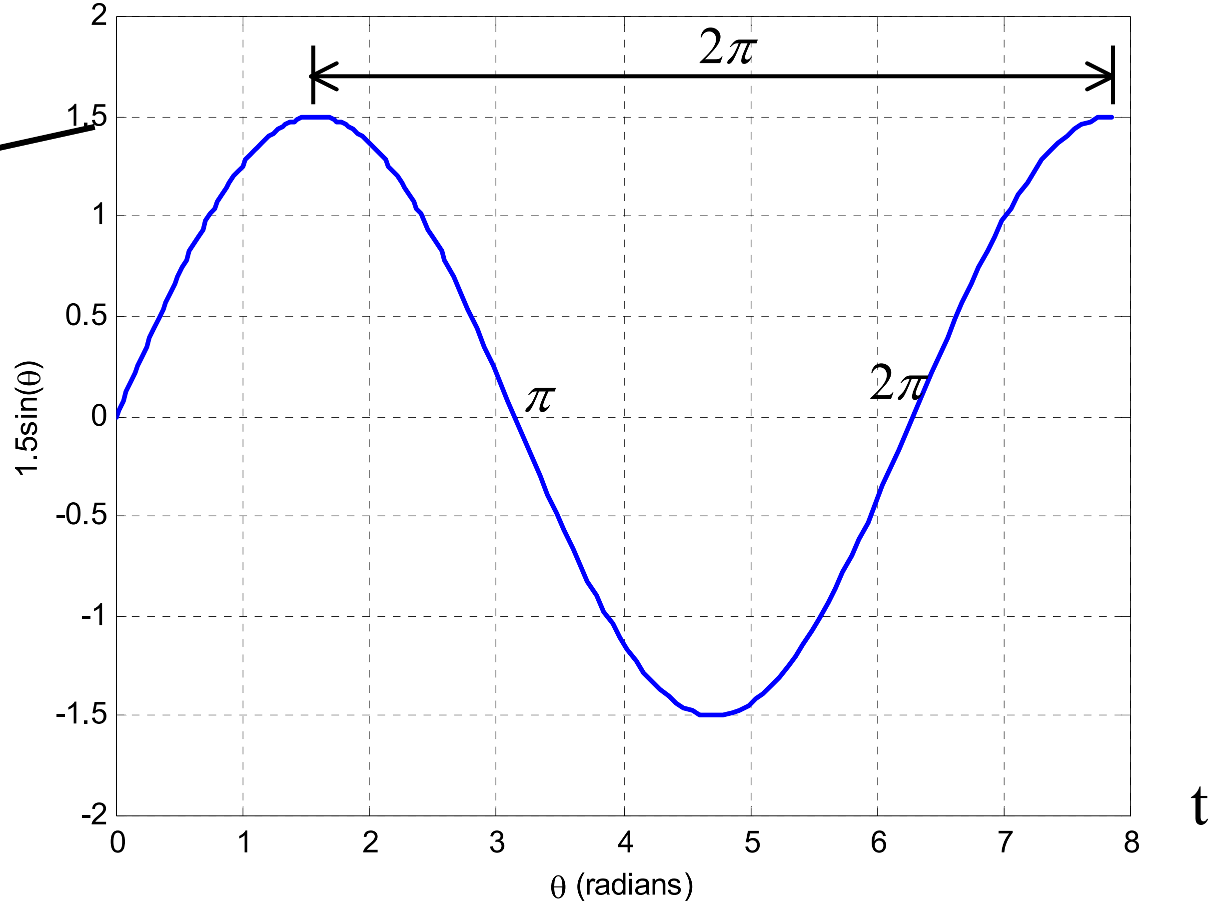
2. All periodic electrical signals can be represented by a sum of sinusoids (Fourier analysis).

→ We will study the steady-state forced response of circuits sourced by sinusoidal driving functions.
(ac steady-state analysis)

Sinusoids

Amplitude

$$A\sin(\theta)$$



Period = 2π

$$A\sin(\theta) = A\sin(\theta + 2\pi)$$

$$A\sin(\theta) = A\sin(\theta + 2k\pi)$$

k is an integer

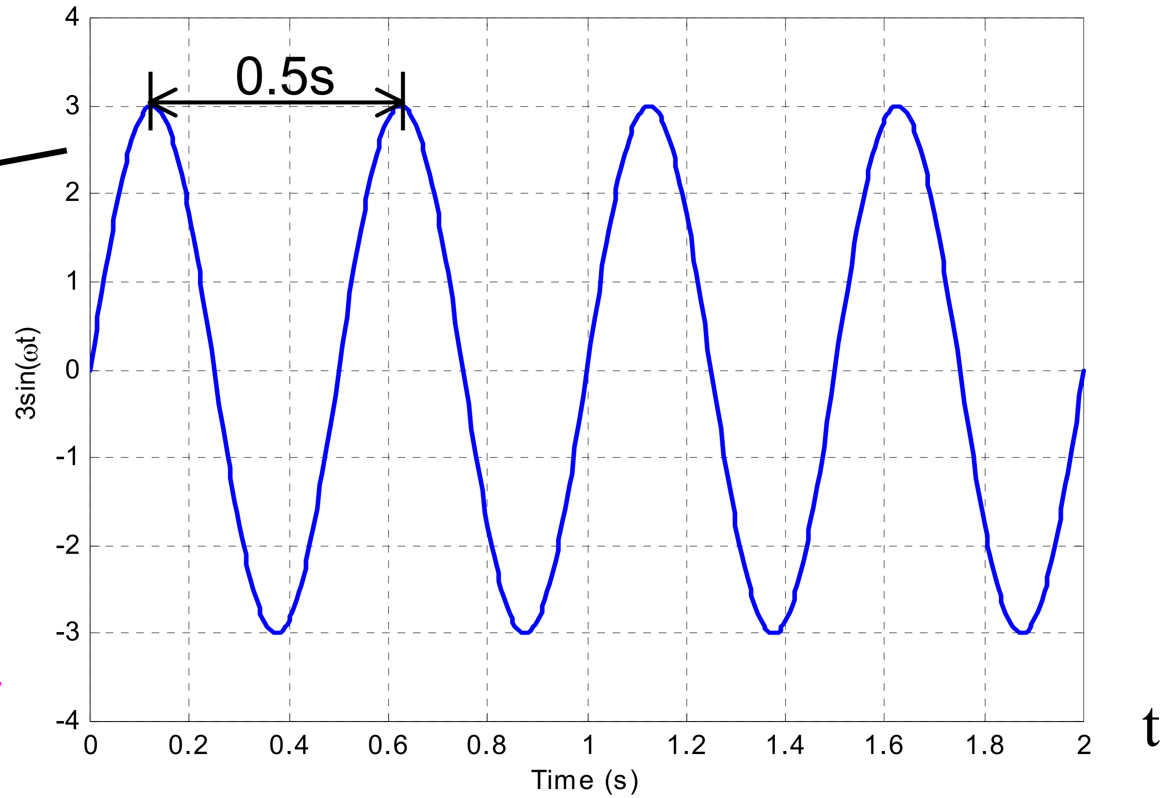
Sinusoids $\theta \rightarrow \omega t$

Amplitude

$$x(t) = A\sin(\omega t)$$

$$\omega = 4\pi \quad \text{rd/s}$$

Angular frequency



Period T of a sine wave: $x(t) = x(t+T)$

$$A\sin(\omega t) = A\sin(\omega(t+T)) = A\sin(\omega t + \omega T)$$

$$\omega T = 2\pi \quad \Rightarrow \quad T = \frac{2\pi}{\omega}$$

Sinusoids

- Period of $x(\theta) = \sin(\theta)$ is 2π radians
- Period of $x(t) = \sin(\omega t)$ is $T = \dfrac{2\pi}{\omega}$ seconds

→ **One cycle takes T seconds.**

→ How many cycles per second?

$$f = \frac{1}{T}$$ Hertz (Hz) or cycles/second

Note: For frequency f in Hertz (Hz), period T in seconds (s) and angular frequency ω in radians/second (rd/s) we have:

$$T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T} \qquad \omega = 2\pi f$$

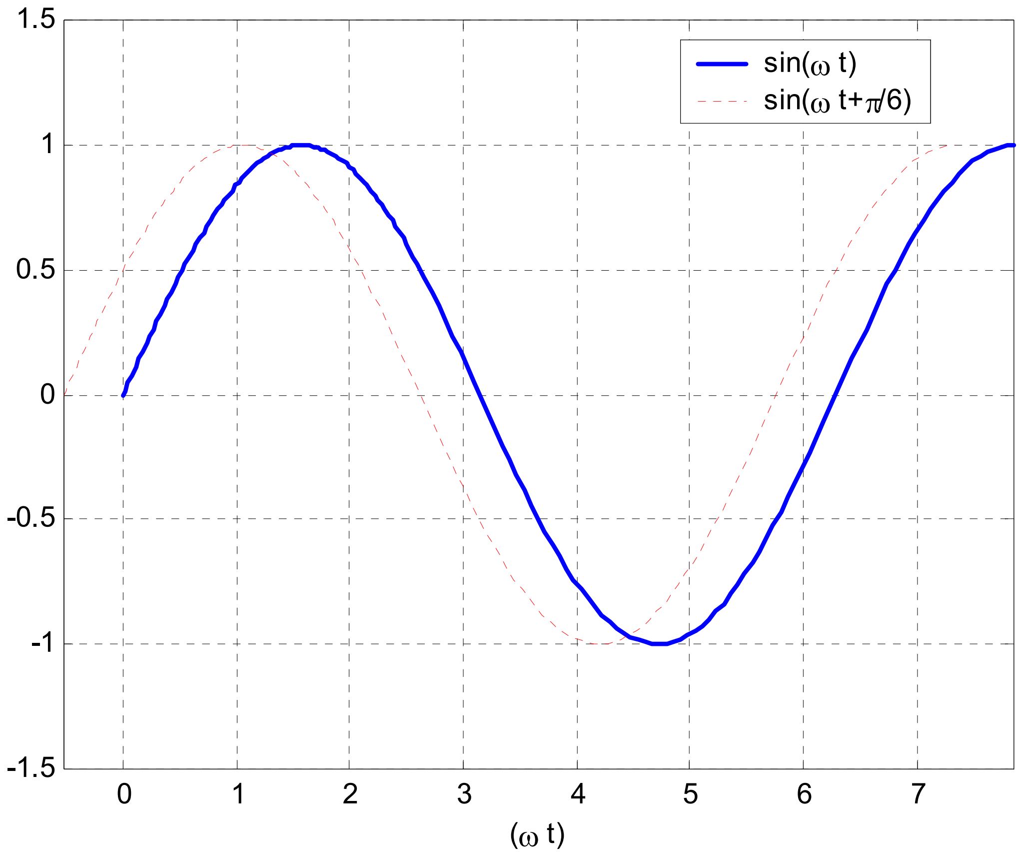
Generalized Sinusoids

Amplitude

$$x(t) = A\sin(\omega t + \theta)$$

Angular frequency

Phase angle



Convention

$A\sin(\omega t)$ **lags** wave $A\sin(\omega t + \theta)$ by θ radians

Phase Angle

By convention:

$$x_1(t) = A_1 \sin(\omega t + \theta_1)$$

$$x_2(t) = A_2 \sin(\omega t + \theta_2)$$

x_1 ***leads*** x_2 by θ_1-θ_2 radians, or
x_2 ***lags*** x_1 by θ_1-θ_2 radians

Also: $\theta_1=\theta_2$ → x_1 and x_2 are "in phase"
$\theta_1\neq\theta_2$ → x_1 and x_2 are "out of phase"

→ We can only compare phase angles if x_1 and x_2 have the same angular frequency ω.
→ The phase angle is often expressed in degrees.
→ When comparing phase angles, express both functions as either sine or cosine with positive amplitude.

Trigonometric Identities (Confirm!)

$$\cos(\omega t) = \sin(\omega t + 90^o) = -\cos(\omega t \pm 180^o)$$

$$\sin(\omega t) = \cos(\omega t - 90^o) = -\sin(\omega t \pm 180^o)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Sum of Sine and Cosine

Since $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$

Then $x(t)=A\sin(\omega t+\theta)$

$$=\underbrace{A\cos(\theta)}_{B_1}\sin(\omega t)+\underbrace{A\sin(\theta)}_{B_2}\cos(\omega t)$$

$$=B_1\sin(\omega t)+B_2\cos(\omega t)$$

$$B_1^2+B_2^2=A^2\left(\cos^2(\theta)+\sin^2(\theta)\right)=A^2 \quad \Rightarrow \quad A=\sqrt{B_1^2+B_2^2}$$

$$\frac{B_2}{B_1}=\frac{\sin(\theta)}{\cos(\theta)}=\tan(\theta) \quad \Rightarrow \quad \theta=\tan^{-1}\left(\frac{B_2}{B_1}\right)$$

Example

$$v_1(t) = 12\sin(377t + 60^o)$$

$$v_2(t) = -6\cos(377t + 30^o)$$

Find frequency and phase angles

Choose to express both as sine with positive amplitude

$$v_1(t) = 12\sin(377t + 60^o)$$

$$v_2(t) = -6\cos(377t + 30^o) = 6\cos(377t + 210^o)$$

$$= 6\sin(377t + 300^o)$$

Phase difference is 60-300=-240^o

→ v_1 leads v_2 by 120^o (Why?)

Also: $f = \dfrac{\omega}{2\pi} = \dfrac{377}{2\pi} = 60$ Hz

Sinusoidal Forcing Function

Recall the form of the ***natural*** response of a linear ***second order circuit***:

$$x_n(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$ → Overdamped

$$x_n(t) = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t}$$ → Critically damped

$$x_n(t) = e^{-\alpha t}\left[B_1 \cos \omega_d t + B_2 \sin \omega_d t\right]$$ → Underdamped

At t=∞ the ***natural response*** goes to zero and we are left with the forced response.

Sinusoidal Forcing Function

Recall the form of the ***forced*** response of a linear second order circuit:

For a sinusoidal forcing function sin(ωt) the forced response has the form:

$$A\sin(\omega t)+B\cos(\omega t)$$

- Therefore, in the steady-state (after all transients have died out):

A linear *second order* circuit with a sinusoidal input will have sinusoidal branch voltages and currents at the same frequency as the input.

Sinusoidal Forcing Function

We can generalize the above conclusion for ***any linear*** circuit:

In the steady-state (after all transients have died out):
A *linear circuit with a sinusoidal input* will have sinusoidal branch voltages and currents at the same frequency as the input.

For example, a circuit in an input of the form:

$$A\sin(\omega t+\theta)$$

will have an output (a solution) in the steady state in the form:

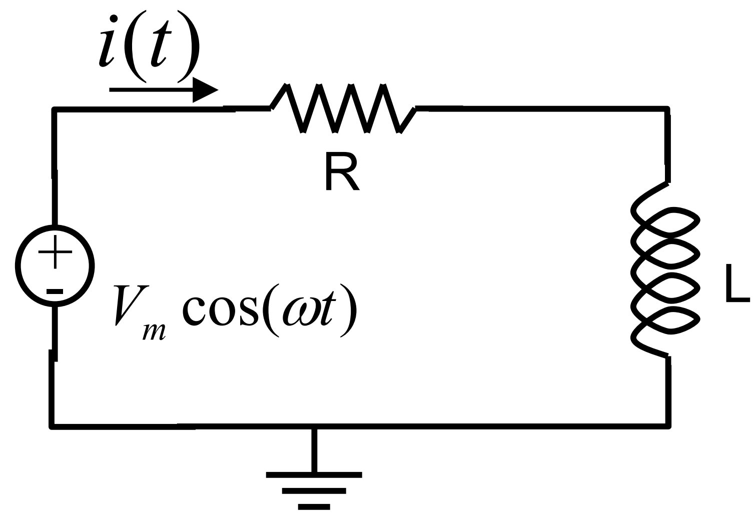
$$B\sin(\omega t+\phi)$$

Need to determine two parameters.

Example

KVL

$$L\frac{di}{dt}+Ri=V_m\cos(\omega t)$$



Assume steady-state (ss):

$$i(t)=A\cos(\omega t+\phi)$$

$$i(t)=A\cos(\phi)\cos(\omega t)-A\sin(\phi)\sin(\omega t)$$

$$i(t)=A_1\cos(\omega t)+A_2\sin(\omega t)$$ → **Substitute into ODE**

$$L\frac{d}{dt}\left(A_1\cos(\omega t)+A_2\sin(\omega t)\right)+R\left(A_1\cos(\omega t)+A_2\sin(\omega t)\right)=V_m\cos(\omega t)$$

Example

$$L\frac{d}{dt}\left(A_1\cos(\omega t)+A_2\sin(\omega t)\right)+R\left(A_1\cos(\omega t)+A_2\sin(\omega t)\right)=V_m\cos(\omega t)$$

$$-L\omega A_1\sin(\omega t)+L\omega A_2\cos(\omega t)+RA_1\cos(\omega t)+RA_2\sin(\omega t)=V_m\cos(\omega t)$$

$$\begin{cases}-L\omega A_1+RA_2=0\\ L\omega A_2+RA_1=V_m\end{cases}\Rightarrow\quad A_1=\frac{RV_m}{R^2+\omega^2L^2}\qquad A_2=\frac{\omega LV_m}{R^2+\omega^2L^2}$$

$$i(t)=\frac{RV_m}{R^2+\omega^2L^2}\cos(\omega t)+\frac{\omega LV_m}{R^2+\omega^2L^2}\sin(\omega t)$$

Example

$$i(t) = \frac{RV_m}{R^2 + \omega^2 L^2}\cos(\omega t) + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin(\omega t)$$

$$i(t) = A\cos(\omega t + \phi)$$

Convert to simpler form

$$i(t) = A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t)$$

Compare to above

$$\tan(\phi) = \frac{A\sin(\phi)}{A\cos(\phi)} = -\frac{\omega L}{R} \quad \Rightarrow \quad \phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$A^2 = \frac{R^2 V_m^2}{\left(R^2 + \omega^2 L^2\right)^2} + \frac{(\omega L)^2 V_m^2}{\left(R^2 + \omega^2 L^2\right)^2} = \frac{V_m^2}{R^2 + \omega^2 L^2}$$

$$A = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$