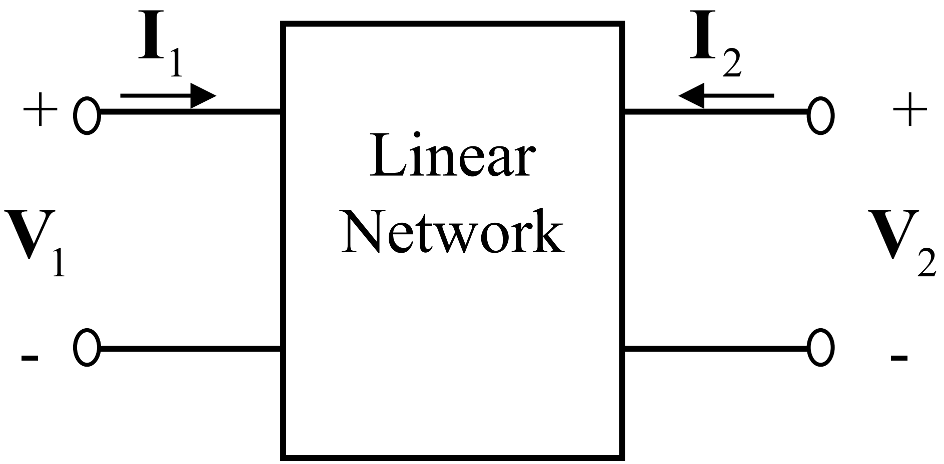
ECSE 210: Circuit Analysis

Lecture #30:

Two-Port Networks

Two-Port Impedance Parameters

→ Commonly referred to as the ***Z-parameters***.



→ Note the voltage polarities, and current directions.

→ Using the principle of superposition we get:

$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 + z_{12}\mathbf{I}_2$$

$$\mathbf{V}_2 = z_{21}\mathbf{I}_1 + z_{22}\mathbf{I}_2$$

Z-Parameters

$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 + z_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = z_{21}\mathbf{I}_1 + z_{22}\mathbf{I}_2$$

$\Rightarrow$

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

- → Two equations describe two-port operation.
- → z_{ij} are **complex** constants of proportionality with units of Ohms (Ω).
- → Once z_{11} , z_{12} , z_{21} , z_{22} are known, the input/output operation of the two-port network is ***completely*** defined.
- → z_{ij} are the **impedance parameters** or Z-parameters.
- → The Z-parameter matrix is:

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

Z-Parameters

$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 + z_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = z_{21}\mathbf{I}_1 + z_{22}\mathbf{I}_2$$

$$\Rightarrow \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

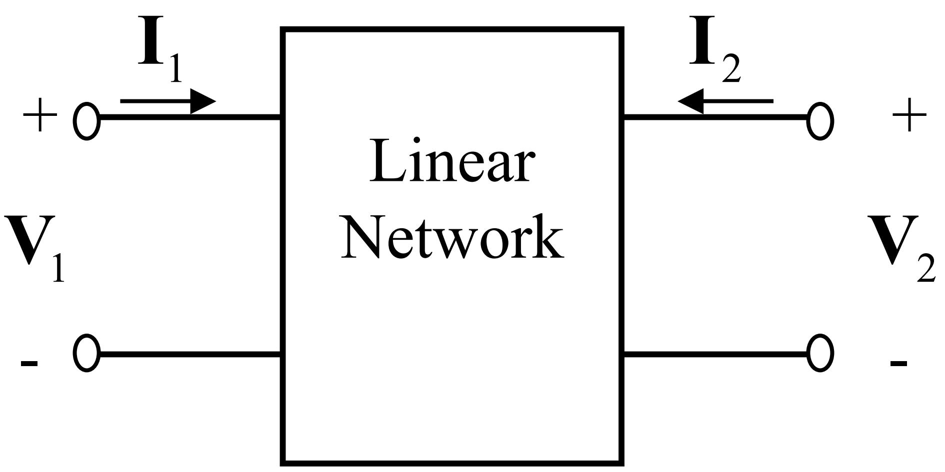
$$\mathbf{I}_2 = 0 \rightarrow \mathbf{V}_1 = z_{11}\mathbf{I}_1 \rightarrow z_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0}$$

$$\mathbf{I}_2 = 0 \rightarrow \mathbf{V}_2 = z_{21}\mathbf{I}_1 \rightarrow z_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0}$$

$$\mathbf{I}_1 = 0 \rightarrow \mathbf{V}_1 = z_{12}\mathbf{I}_2 \rightarrow z_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0}$$

$$\mathbf{I}_1 = 0 \rightarrow \mathbf{V}_2 = z_{22}\mathrm{I}_2 \rightarrow z_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0}$$

Z-Parameters vs. Y-Parameters



$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1}$$

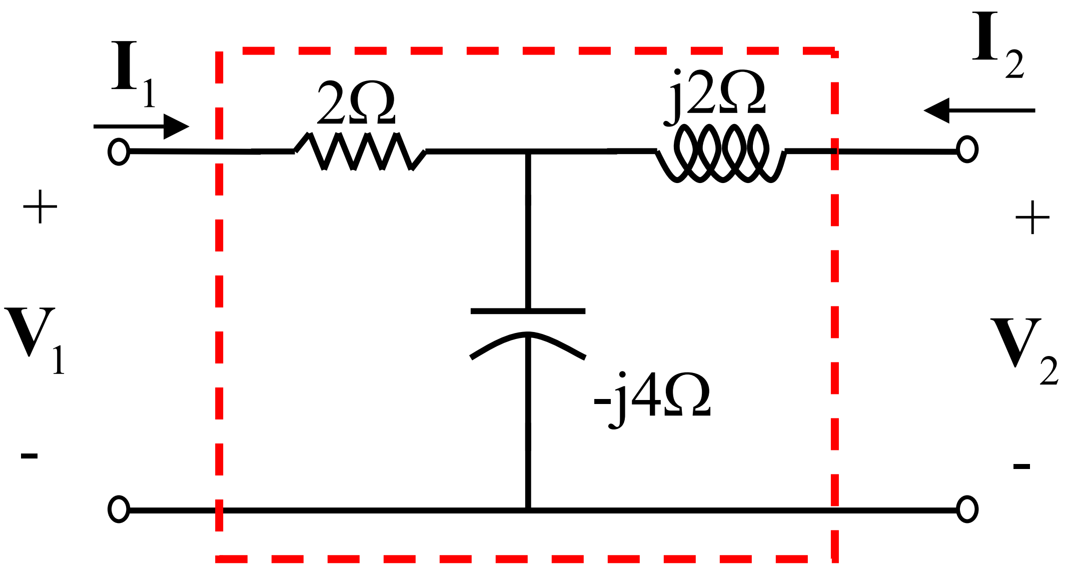
$$\mathbf{Z} = \mathbf{Y}^{-1}$$

Example

→ Z-parameters can be found "experimentally".

$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 + z_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = z_{21}\mathbf{I}_1 + z_{22}\mathbf{I}_2$$

$$\Rightarrow \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

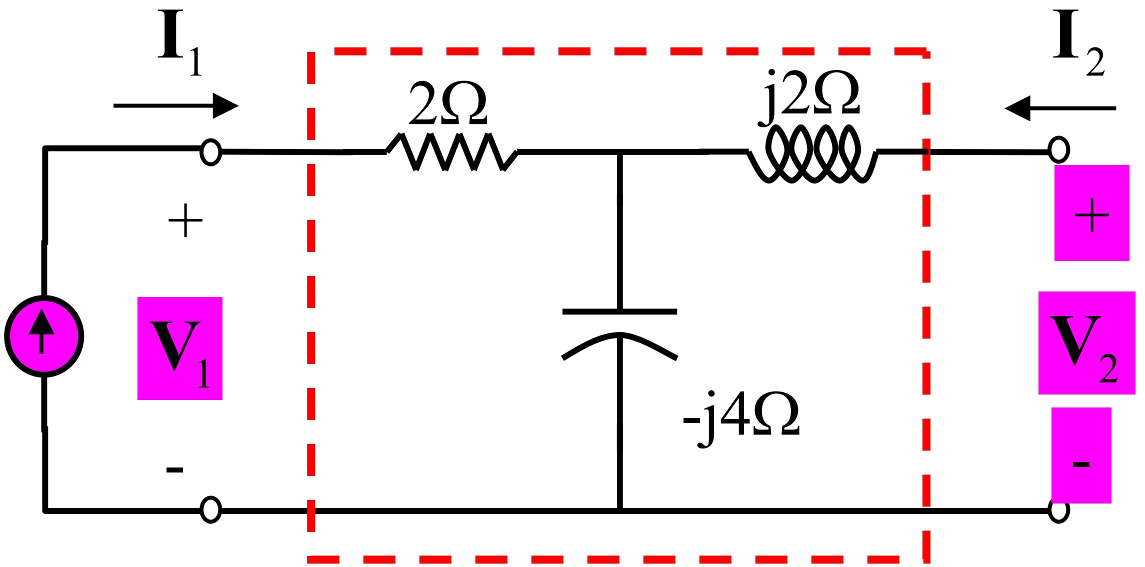


Example

→ **Open circuit port 2:**

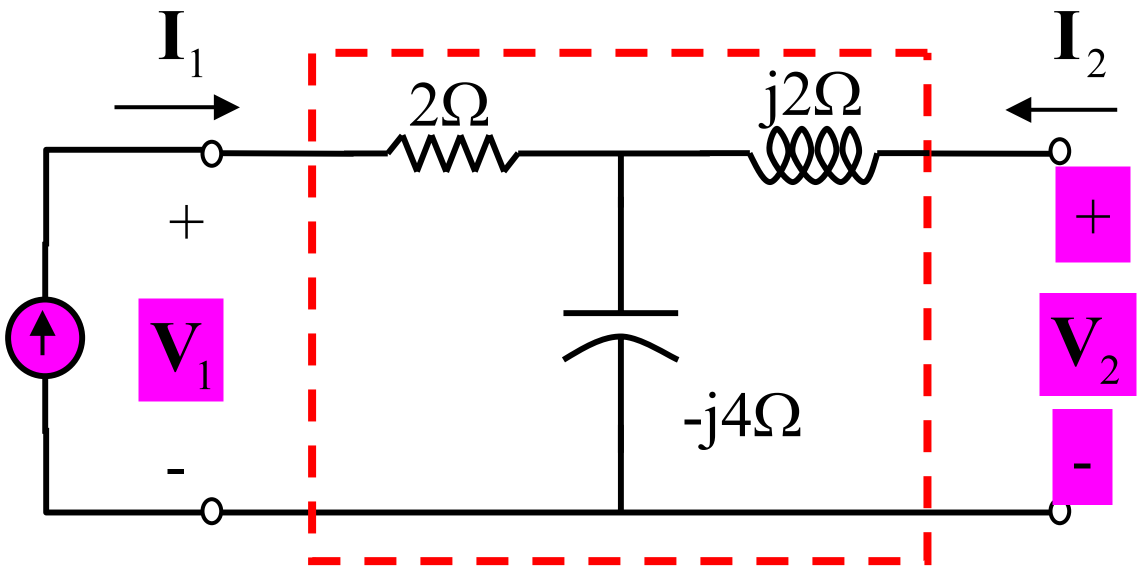
$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 + z_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = z_{21}\mathbf{I}_1 + z_{22}\mathbf{I}_2$$

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$



$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 \qquad \mathbf{V}_2 = z_{21}\mathbf{I}_1$$

Example



$$\mathbf{V}_1 = (2-4j)\mathbf{I}_1$$

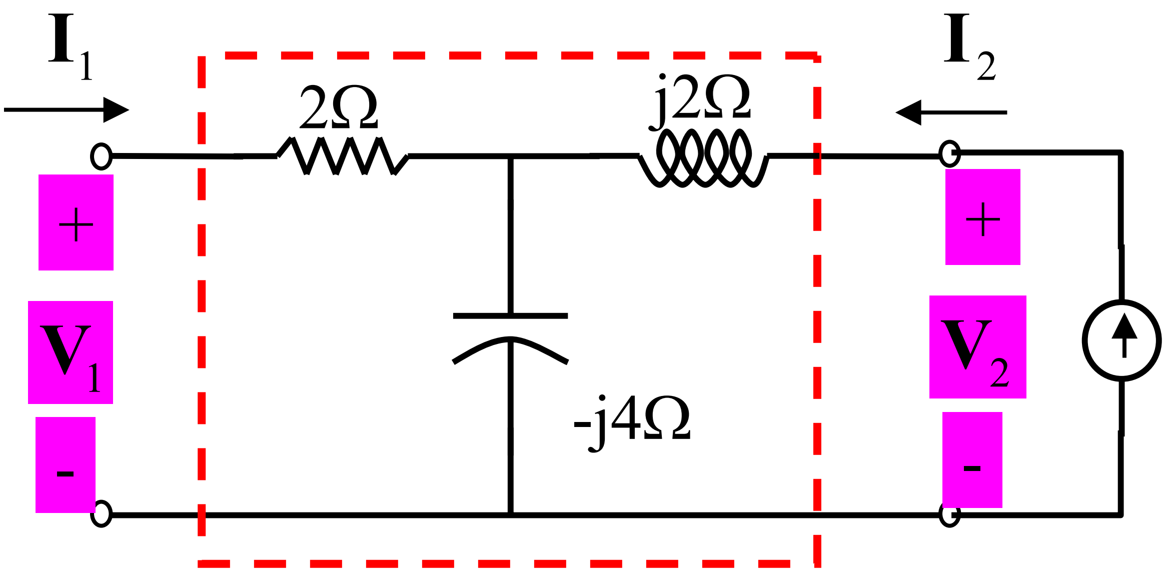
$$z_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = 2-4j\Omega$$

$$\mathbf{V}_2 = \frac{-4j}{2-4j}\mathbf{V}_1 = \frac{-4j}{2-4j}(2-4j)\mathbf{I}_1$$

$$z_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = -4j\Omega$$

Example

→ **Open circuit port 1:**



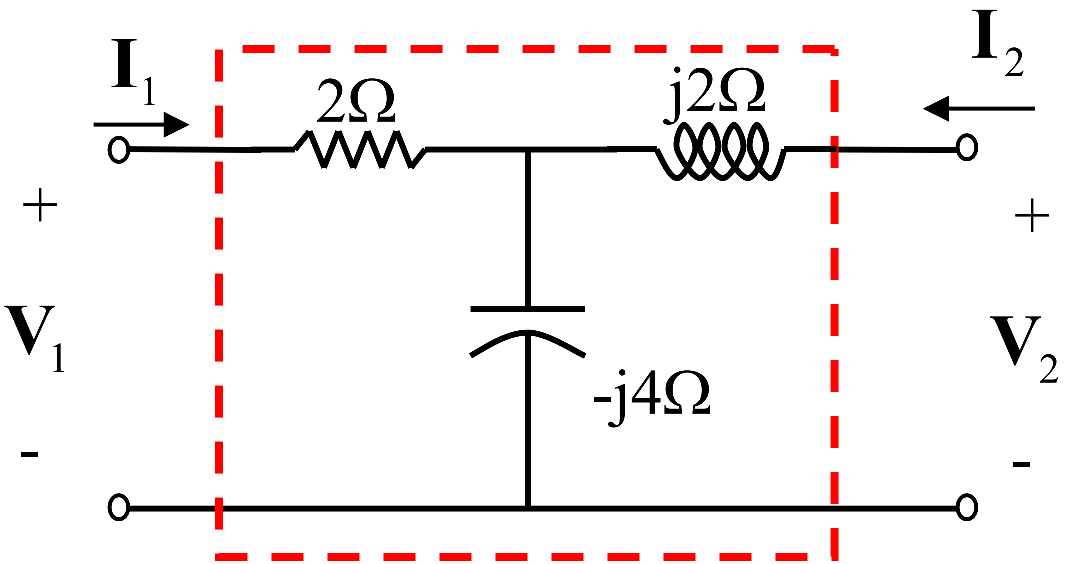
$$\mathbf{V}_2 = (2j - 4j)\mathbf{I}_2 = -2j\mathbf{I}_2$$

$$z_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = -2j\Omega$$

$$\mathbf{V}_1 = \frac{-4j}{2j-4j}\mathbf{V}_2 = \frac{-4j}{-2j}(-2j)\mathbf{I}_2$$

$$z_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = -4j\Omega$$

Example

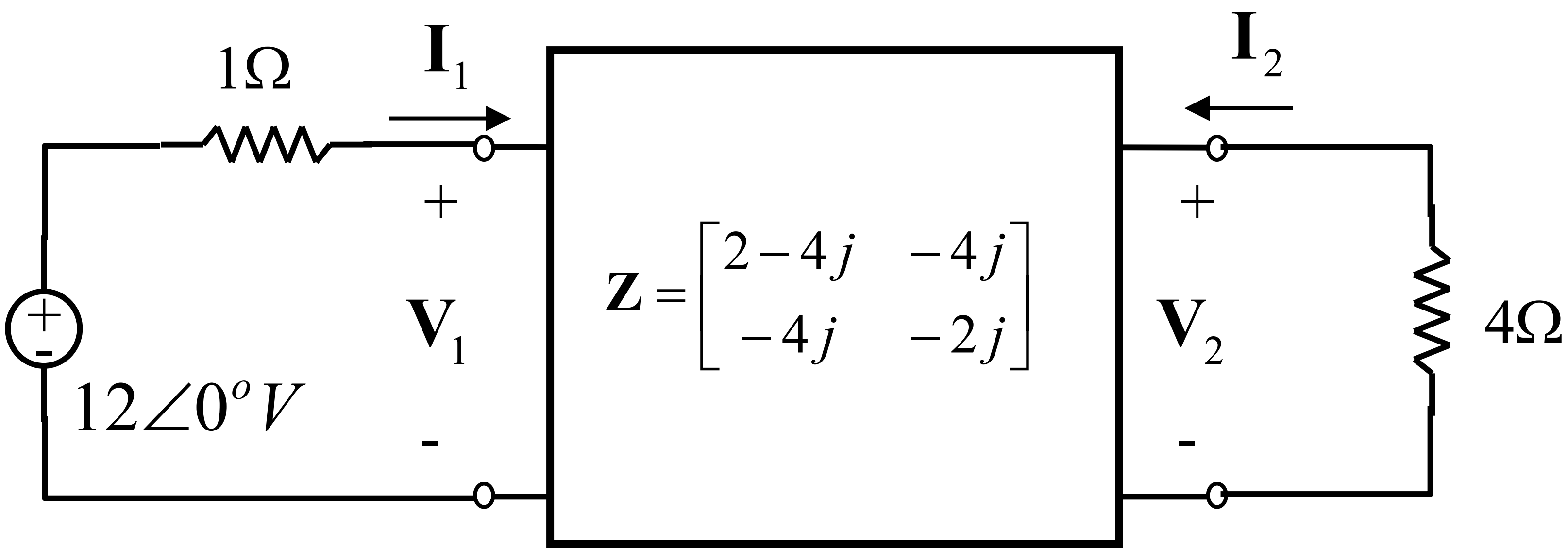


$$\mathbf{V}_1 = (2-4j)\mathbf{I}_1 - 4j\mathbf{I}_2$$
$$\mathbf{V}_2 = -4j\mathbf{I}_1 - 2j\mathbf{I}_2$$

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} = \begin{bmatrix}2-4j & -4j\\ -4j & -2j\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}$$

Example 2

Find the current through the 4Ω resistor.



$$\left.\begin{aligned} \mathbf{V}_1 &= 12\angle 0 - (1\Omega)\mathbf{I}_1 \\ \mathbf{V}_2 &= -(4\Omega)\mathbf{I}_2 \end{aligned}\right\}$$

Boundary conditions due to terminations.

Example 2

Z-parameter equations:

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 2-4j & -4j \\ -4j & -2j \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \Rightarrow \begin{matrix} \mathbf{V}_1 = (2-4j)I_1 - 4j\mathbf{I}_2 \\ \mathbf{V}_2 = -4j\mathbf{I}_1 - 2j\mathbf{I}_2 \end{matrix}$$

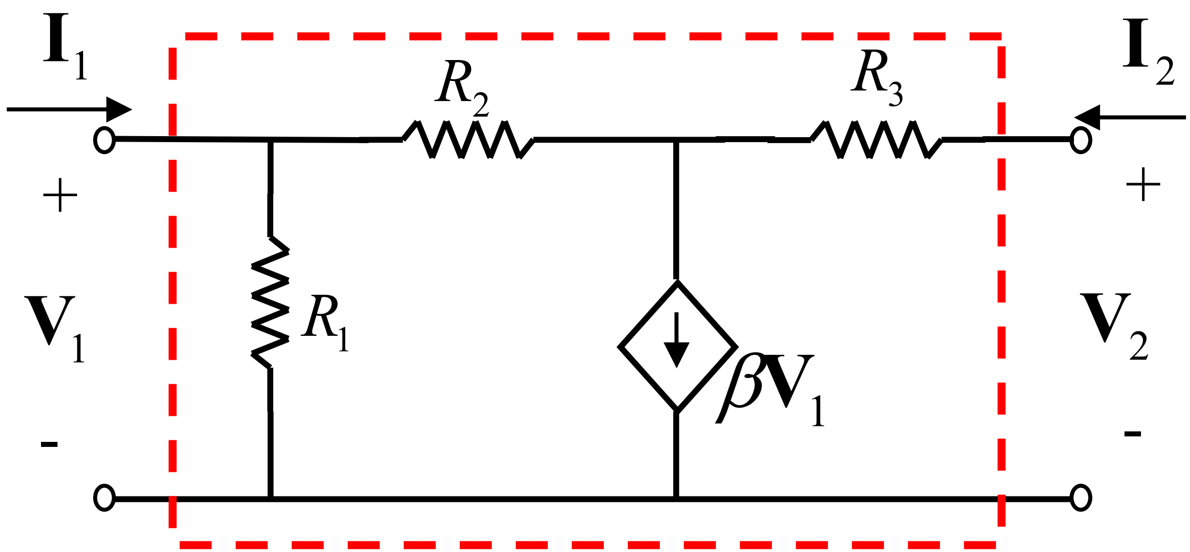
Boundary conditions:

$\mathbf{V}_1 = 12\angle 0 - (1\Omega)\mathbf{I}_1$ $\quad$ $\mathbf{V}_2 = -(4\Omega)\mathbf{I}_2$ $\quad$ (Eliminates $\mathbf{V}_1$ and $\mathbf{V}_2$)

$$\Rightarrow \begin{cases} (3-4j)\mathbf{I}_1 - 4j\mathbf{I}_2 = 12\angle 0^o \\ -4j\mathbf{I}_1 + (4-2j)\mathbf{I}_2 = 0 \end{cases} \Rightarrow \mathbf{I}_2 = 1.61\angle 137.7^o\ A$$

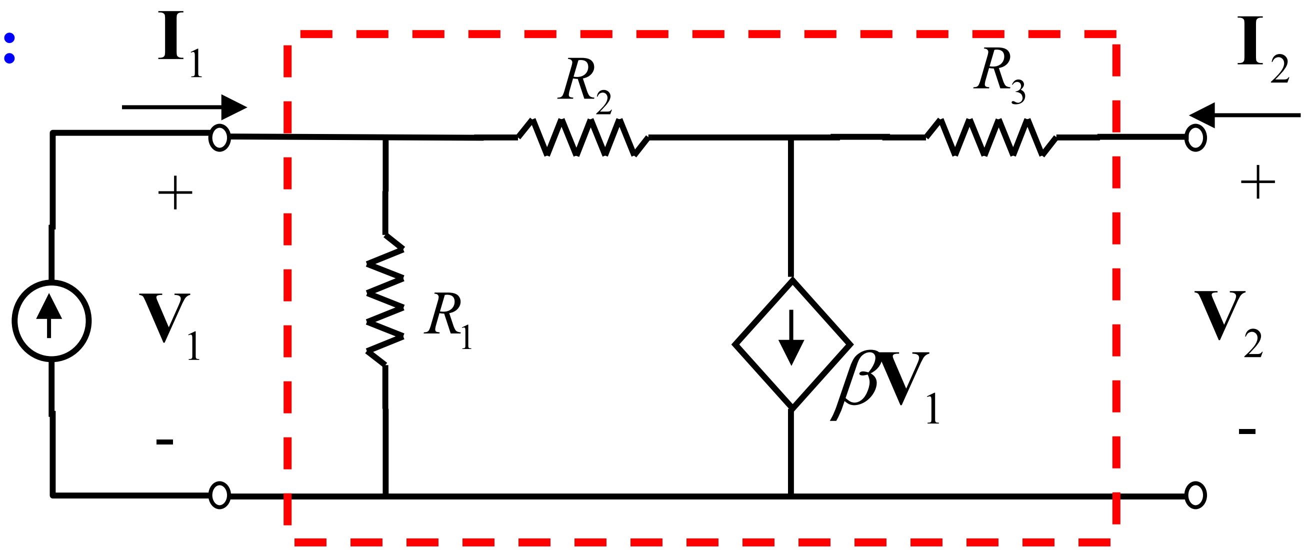
Example 3

Find the impedance (Z) parameters of the circuit below:



Example 3

Open-circuit port 2:



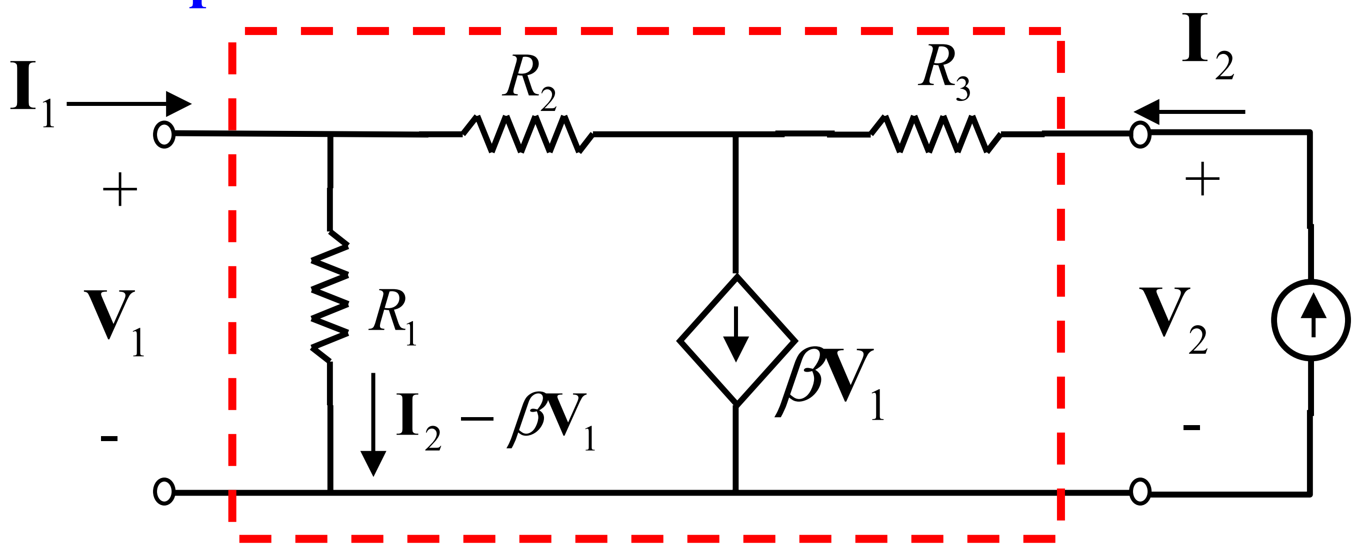
$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{R_1} + \beta \mathbf{V}_1 \quad \Rightarrow \quad z_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{R_1}{1+\beta R_1}\,\Omega \quad ❶$$

$$\mathbf{V}_2 = \mathbf{V}_1 - R_2(\beta \mathbf{V}_1) = (1-\beta R_2)\frac{R_1}{1+\beta R_1}\mathbf{I}_1$$

$$z_{21} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_1}\right|_{\mathbf{I}_2=0} = \frac{R_1(1-\beta R_2)}{1+\beta R_1}\,\Omega \quad ❷$$

Example 3

Open-circuit port 1:



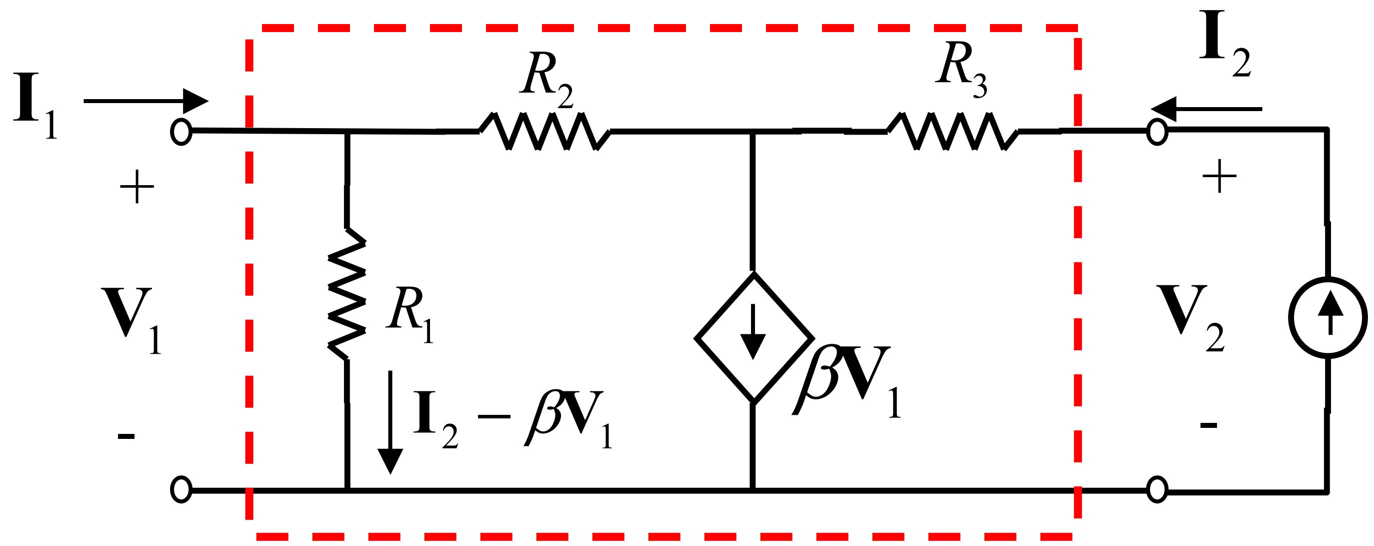
$$V_2 = R_3 I_2 + (R_1 + R_2)(I_2 - \beta V_1) = (R_1 + R_2 + R_3) I_2 - (R_1 + R_2)\beta V_1$$

$$V_2 = (R_1 + R_2 + R_3) I_2 - (R_1 + R_2)\beta \frac{R_1}{R_1 + R_2}(V_2 - I_2 R_3)$$

$$V_2 (1 + \beta R_1) = (R_1 + R_2 + R_3 + \beta R_1 R_3) I_2$$

Example 3

Open-circuit port 1 (cont'd):

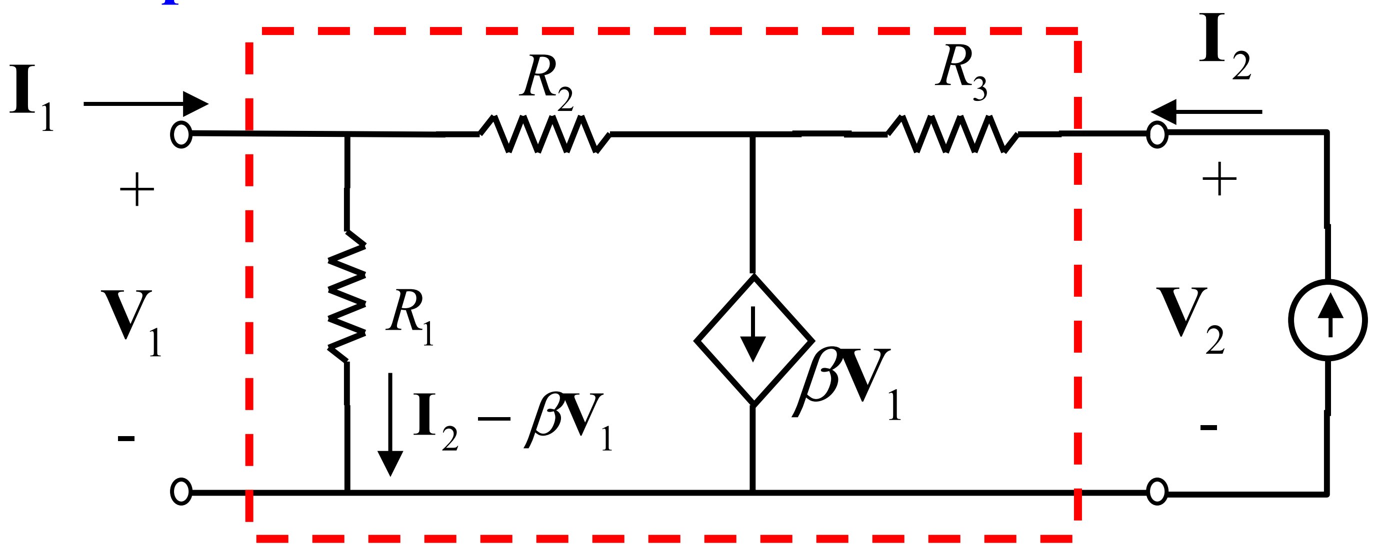


$$\mathbf{V}_2(1+\beta R_1) = (R_1 + R_2 + R_3 + \beta R_1 R_3)\mathbf{I}_2$$

$$z_{22} = \left.\frac{\mathbf{V}_2}{\mathbf{I}_2}\right|_{\mathbf{I}_1=0} = \frac{(R_1 + R_2 + R_3 + \beta R_1 R_3)}{(1+\beta R_1)} \qquad ❸$$

Example 3

Open-circuit port 1:

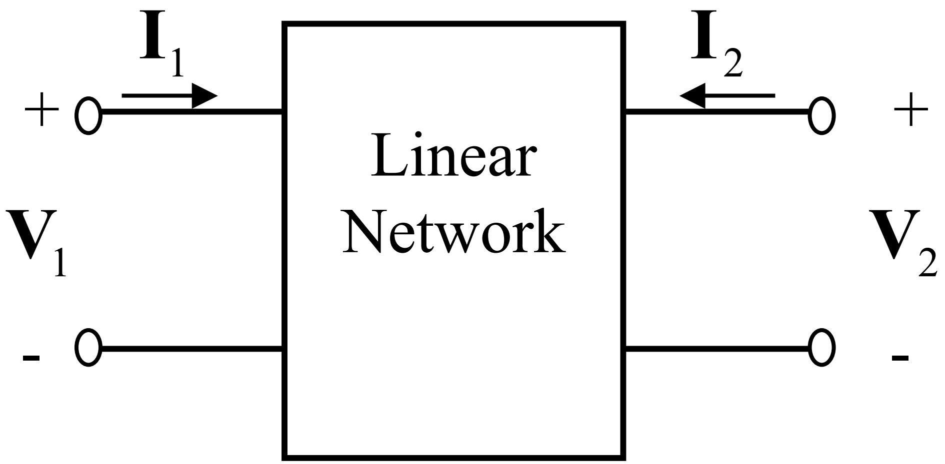


$$V_1 = R_1(I_2 - \beta V_1)$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{R_1}{1+\beta R_1}\,\Omega \qquad ❹$$

Two-Port Hybrid Parameters

→ Commonly referred to as the **H-*parameters***.



→ Note the voltage polarities and current directions.

→ Using the principle of superposition we get:

$$\mathbf{V}_1 = h_{11}\mathbf{I}_1 + h_{12}\mathbf{V}_2$$

$$\mathbf{I}_2 = h_{21}\mathbf{I}_1 + h_{22}\mathbf{V}_2$$

H-Parameters

$$\mathbf{V}_1 = h_{11}\mathbf{I}_1 + h_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = h_{21}\mathbf{I}_1 + h_{22}\mathbf{V}_2$$

$$\Rightarrow \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

- → **Two equations describe two-port operation.**
- → h_{ij} are complex constants of proportionality with units of Ohms (Ω), Siemens (S) or no units.
- → Common in transistor models.
- → Once h_{11} , h_{12} , h_{21} , h_{22} are known, the input/output operation of the two-port network is ***completely*** defined.
- → h_{ij} are the hybrid parameters or H-parameters.
- → The H-parameter matrix is:

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

H-Parameters

$$\mathbf{V}_1 = h_{11}\mathbf{I}_1 + h_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = h_{21}\mathbf{I}_1 + h_{22}\mathbf{V}_2$$

$$\Rightarrow \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

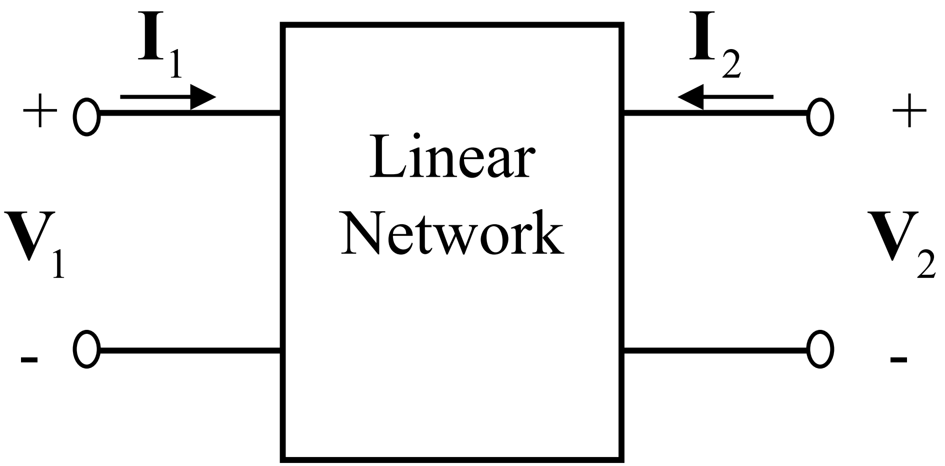
$$\mathbf{V}_2 = 0 \quad \rightarrow \quad \mathbf{V}_1 = h_{11}\mathbf{I}_1 \quad \rightarrow \quad h_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}$$

$$\mathbf{V}_2 = 0 \quad \rightarrow \quad \mathbf{I}_2 = h_{21}\mathbf{I}_1 \quad \rightarrow \quad h_{21} = \left.\frac{\mathbf{I}_2}{\mathbf{I}_1}\right|_{\mathbf{V}_2=0}$$

$$\mathbf{I}_1 = 0 \quad \rightarrow \quad \mathbf{V}_1 = h_{12}\mathbf{V}_2 \quad \rightarrow \quad h_{12} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0}$$

$$\mathbf{I}_1 = 0 \quad \rightarrow \quad \mathbf{I}_2 = h_{22}\mathbf{V}_2 \quad \rightarrow \quad h_{22} = \left.\frac{\mathbf{I}_2}{\mathbf{V}_2}\right|_{\mathbf{I}_1=0}$$

Inverse Hybrid Parameters



$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}^{-1}$$

$$\mathbf{G} = \mathbf{H}^{-1}$$

Hybrid/Inverse Hybrid Parameters

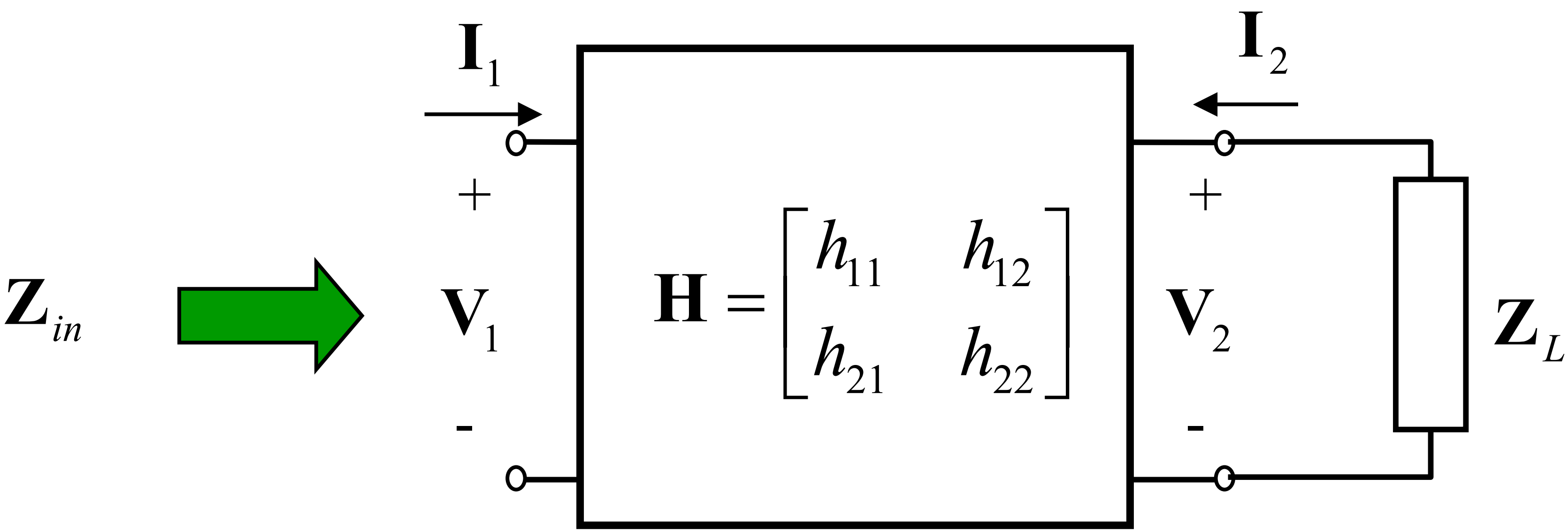
➔ **How to find Hybrid/Inverse Hybrid parameters.**

→ Given hybrid parameters, calculate the inverse hybrid parameters by taking the inverse of the matrix and vice versa.

or

→ Calculate parameters directly using open/short circuit tests.

Example



Calculate the input impedance

$$\mathbf{Z}_{in} = \frac{\mathbf{V}_1}{\mathbf{I}_1}$$

Example

$$\left.\begin{aligned}\mathbf{V}_1 &= h_{11}\mathbf{I}_1 + h_{12}\mathbf{V}_2 \\ \mathbf{I}_2 &= h_{21}\mathbf{I}_1 + h_{22}\mathbf{V}_2\end{aligned}\right\}$$

Hybrid parameters equations

$$\mathbf{V}_2 = -\mathbf{I}_2\mathbf{Z}_L$$

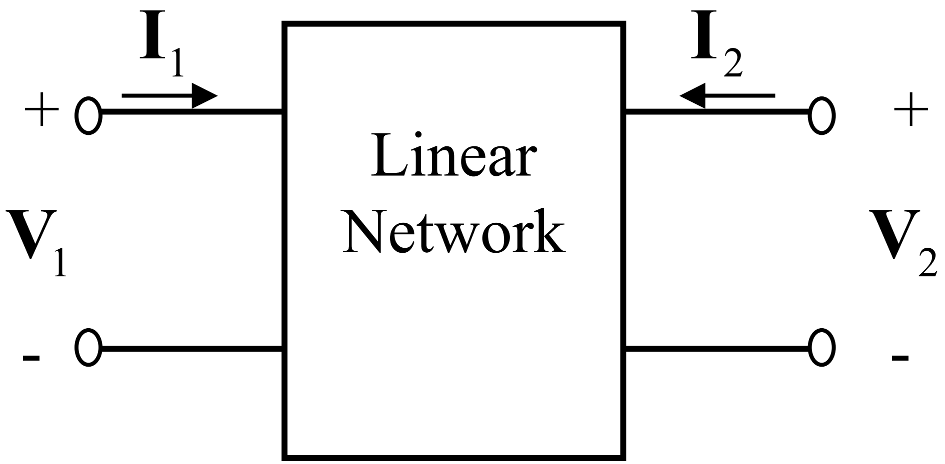
Terminal condition

$$\left\{\begin{aligned}\mathbf{V}_1 &= h_{11}\mathbf{I}_1 - h_{12}(\mathbf{I}_2\mathbf{Z}_L) \\ \mathbf{I}_2 &= h_{21}\mathbf{I}_1 - h_{22}(\mathbf{I}_2\mathbf{Z}_L)\end{aligned}\right.$$

$$\Rightarrow \quad \mathbf{Z}_{in} = h_{11} - \frac{h_{12}h_{21}\mathbf{Z}_L}{1 + h_{22}\mathbf{Z}_L}$$

Two-Port Transmission Parameters

→ Commonly referred to as the **ABCD-*parameters***.



→ Note the voltage polarities and current directions.

→ Using the principle of superposition we get:

$$\mathbf{V}_1 = A\mathbf{V}_2 - B\mathbf{I}_2$$

$$\mathbf{I}_1 = C\mathbf{V}_2 - D\mathbf{I}_2$$

ABCD- Parameters

$$\mathbf{V}_1 = A\mathbf{V}_2 - B\mathbf{I}_2$$
$$\mathbf{I}_1 = C\mathbf{V}_2 - D\mathbf{I}_2$$

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

- → Two equations describe two-port operation.
- → *A, B, C and D* are ***complex*** constants of proportionality with units of Ohms (Ω), Siemens (S), or no units.
- → Common in transmission line models.
- → Once *A, B, C and D* are known, the input/output operation of the two-port network is ***completely*** defined.
- → The transmission parameter matrix is:

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$