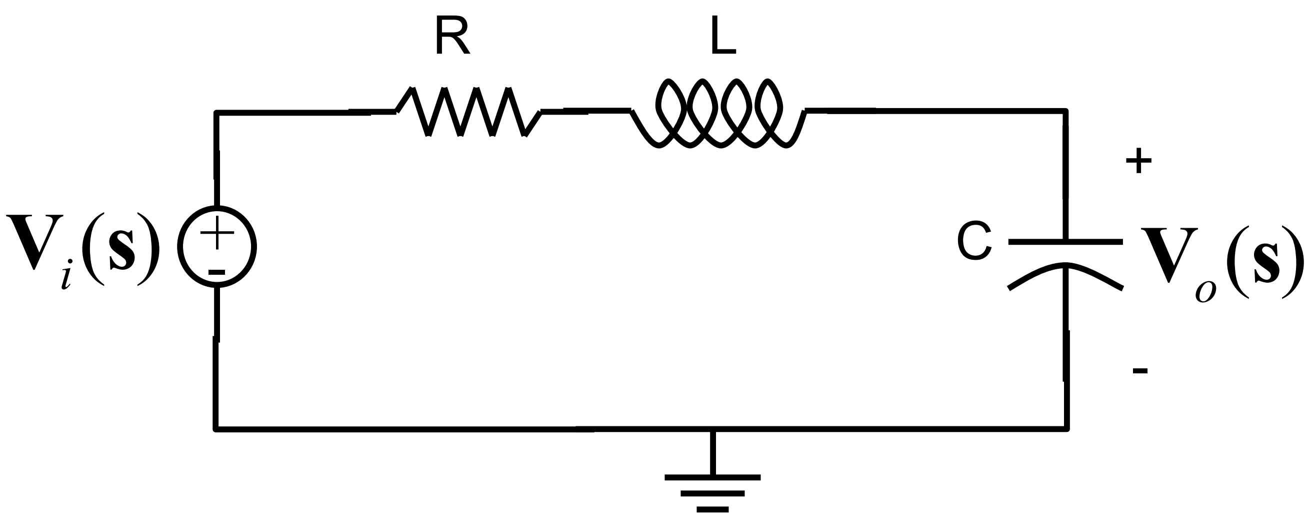
ECSE 210: Circuit Analysis

Lecture #26:

Resonant Circuits

Poles and Circuit Dynamics



$$\mathbf{V}_o(\mathbf{s}) = \frac{1/\mathbf{s}C}{R + \mathbf{s}L + 1/\mathbf{s}C}\mathbf{V}_i(\mathbf{s}) = \frac{1}{LC\mathbf{s}^2 + RC\mathbf{s} + 1}\mathbf{V}_i(\mathbf{s})$$

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{1/LC}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}}$$

Poles and Circuit Dynamics

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{1/LC}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}} = \frac{\omega_o^2}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

$$\zeta = \frac{R/L}{2\omega_o}$$

so that

$$\zeta = \frac{R/L}{2\omega_o} = \frac{R}{L}\frac{1}{2}\sqrt{LC} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Poles and Circuit Dynamics

$$\mathbf{H(s)} = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{\omega_o^2}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$

If $\zeta < 1 \Rightarrow p_{1,2} = -\zeta\omega_o \pm j\omega_o\sqrt{1-\zeta^2} = -\alpha \pm j\omega_d$

$$\omega_d = \omega_o\sqrt{1-\zeta^2}$$

For $\zeta < \frac{1}{\sqrt{2}} \Rightarrow$ **the amplitude peaks at** $\omega_r = \omega_o\sqrt{1-2\zeta^2}$

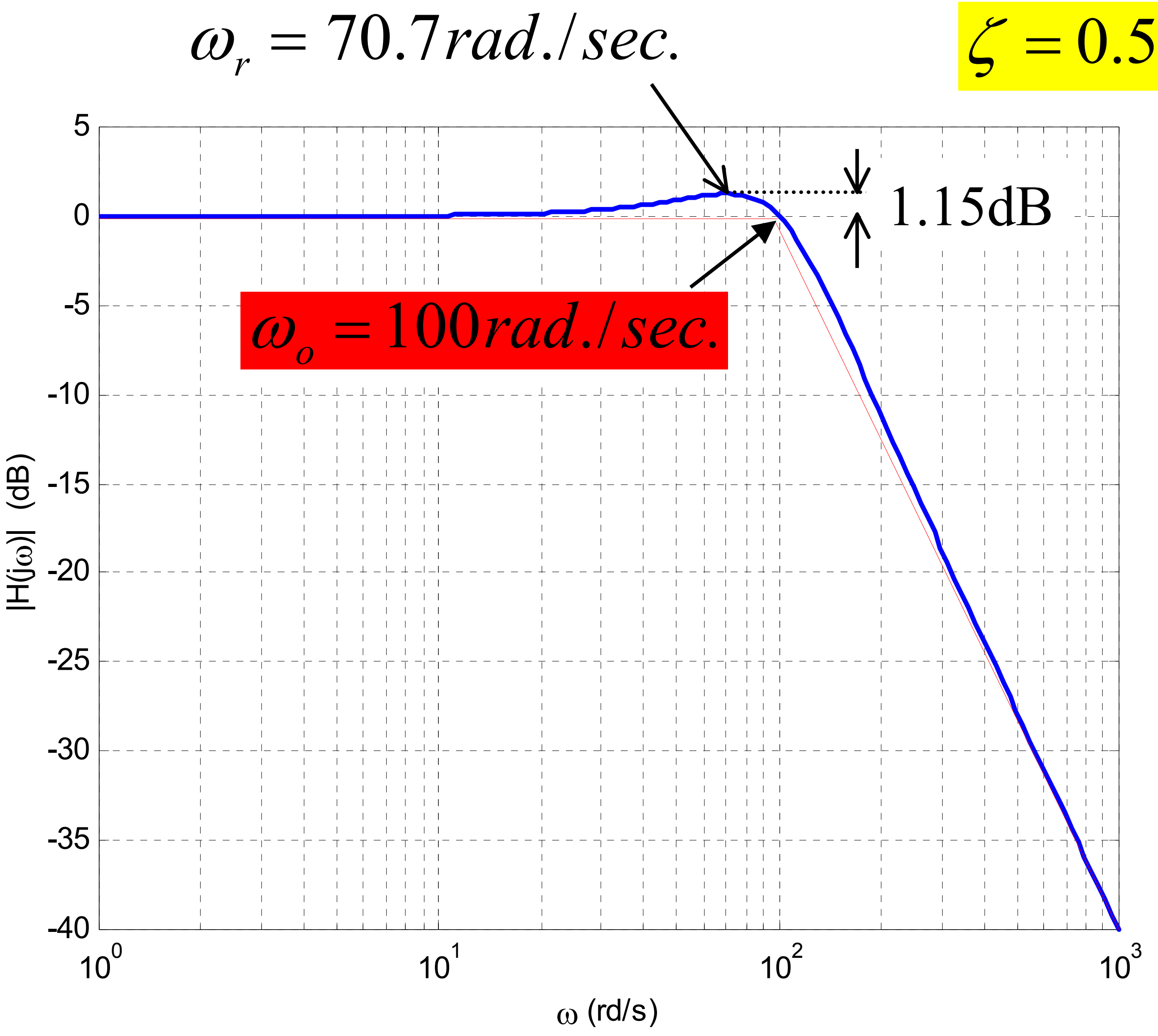
→ ω_o **is the undamped natural frequency (from the circuit).**

→ ω_d **is the damped natural frequency.**

→ ω_r is the resonant frequency.

Poles and Frequency Response

$\zeta = 0.5 < 0.707$



Poles and Natural Response

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{\omega_o^2}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$

$\zeta < 1$ ⇒ $$p_{1,2} = -\zeta\omega_o \pm j\omega_o\sqrt{1-\zeta^2} = -\alpha \pm j\omega_d$$

$$\omega_d = \omega_o\sqrt{1-\zeta^2}$$

Natural response:

$$v_n(t) = A_n e^{-\alpha t}\cos(\omega_d t + \phi)$$

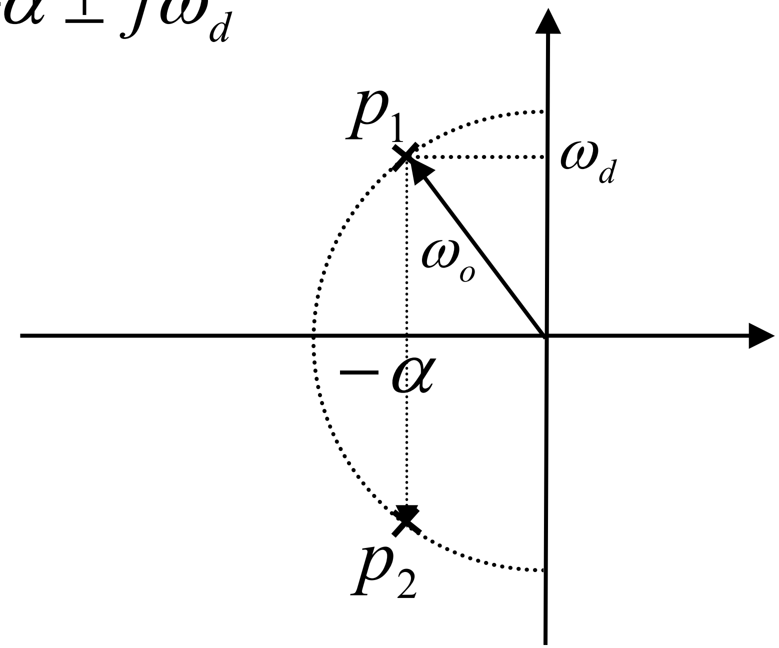
→ Damped natural frequency

Poles and Natural Response

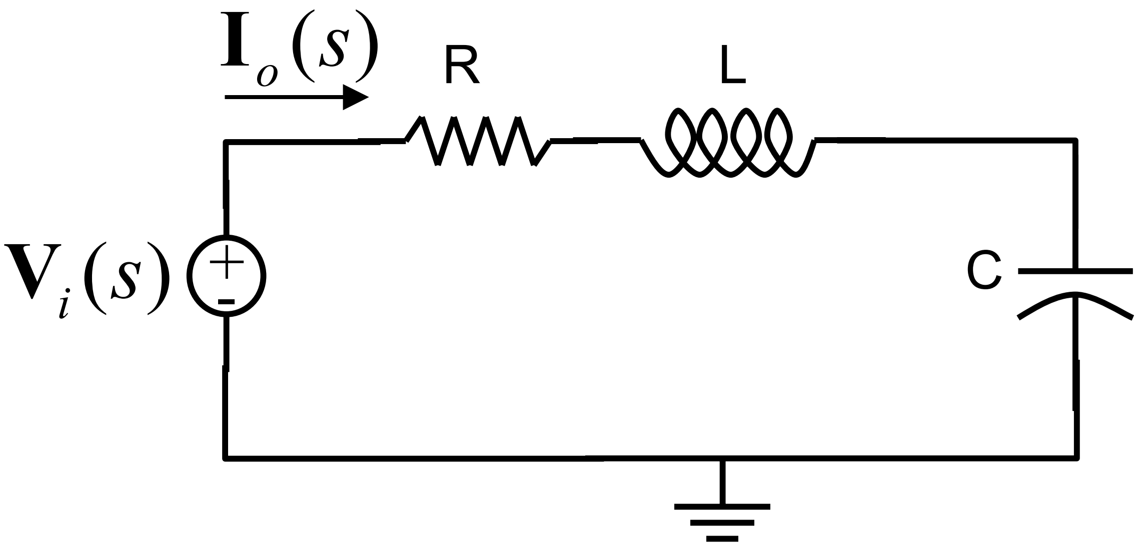
$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \frac{1/LC}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}} = \frac{\omega_o^2}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2}$$

$$p_{1,2} = -\zeta\omega_o \pm j\omega_o\sqrt{1-\zeta^2} = -\alpha \pm j\omega_d$$

complex poles



Series Resonant Circuit



$$\mathbf{H}(s) = \frac{\mathbf{I}_o(s)}{\mathbf{V}_i(s)} = \mathbf{Y}(s) = \frac{1}{R + sL + \frac{1}{sC}} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$= \frac{N(s)}{s^2 + 2\zeta\omega_o s + \omega_o^2}$$

Note that *D(s)* is second order ➔

Resonant Circuits

$$\mathbf{H(s)} = \frac{N(\mathbf{s})}{\mathbf{s}^2 + 2\zeta\omega_o\mathbf{s} + \omega_o^2} = \frac{\frac{1}{L}\mathbf{s}}{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}}$$

$$\omega_o = \sqrt{\frac{1}{LC}} \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Remember: $\omega_r = \omega_o\sqrt{1-2\zeta^2}$.

$$p_{1,2} = -\zeta\omega_o \pm j\omega_o\sqrt{1-\zeta^2}$$

How does the numerator affect the response $I_o(s)$ for a given input $V_i(s)$?

What is the resonant frequency, ω_r?

Resonant Circuits

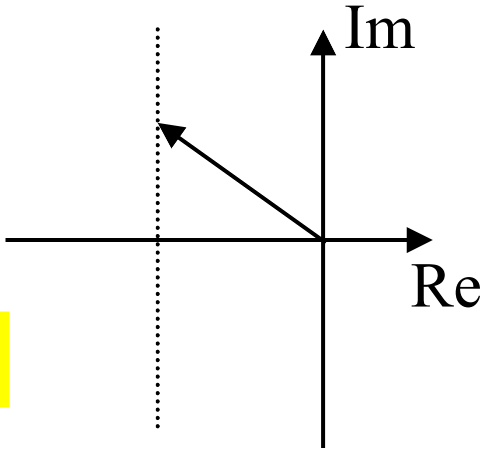
ω_r, the resonant frequency, is defined as the frequency at which $|H(j\omega)|$ is a maximum (or minimum).

$$\mathbf{H}(s) = \frac{\mathbf{I}_o(\mathbf{s})}{\mathbf{V}_i(\mathbf{s})} = \mathbf{Y}(\mathbf{s}) = \frac{N(\mathbf{s})}{R + \mathbf{s}L + \frac{1}{\mathbf{s}C}} = \frac{N(j\omega)}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

Maximum occurs when:

$$\omega L - \frac{1}{\omega C} = 0$$

$$\omega_r = \sqrt{\frac{1}{LC}} \longrightarrow$$ Same as ω_o in this case.



Resonant Circuits

It is often difficult to find the exact frequency at which the maximum response occurs for a given circuit. There are two common approximations that work reasonably well for lightly damped circuits:

- → The frequency at which the impedance $\mathbf{Z(s)=Y^{-1}(s)}$ is real is taken to be the resonant frequency.
- → The undamped natural frequency ω_o is taken to be the resonant frequency.

Recall: Quality Factor

The quality factor of a circuit is defined in terms of energy:

$$Q = 2\pi \frac{\text{total energy stored}}{\text{energy dissipated per cycle}}$$

The resonant Q is the Q at the resonant frequency.

$$Q_o = 2\pi \frac{\textit{maximum} \text{ total energy stored}}{\text{energy dissipated per cycle}}$$

For the series resonant circuit:

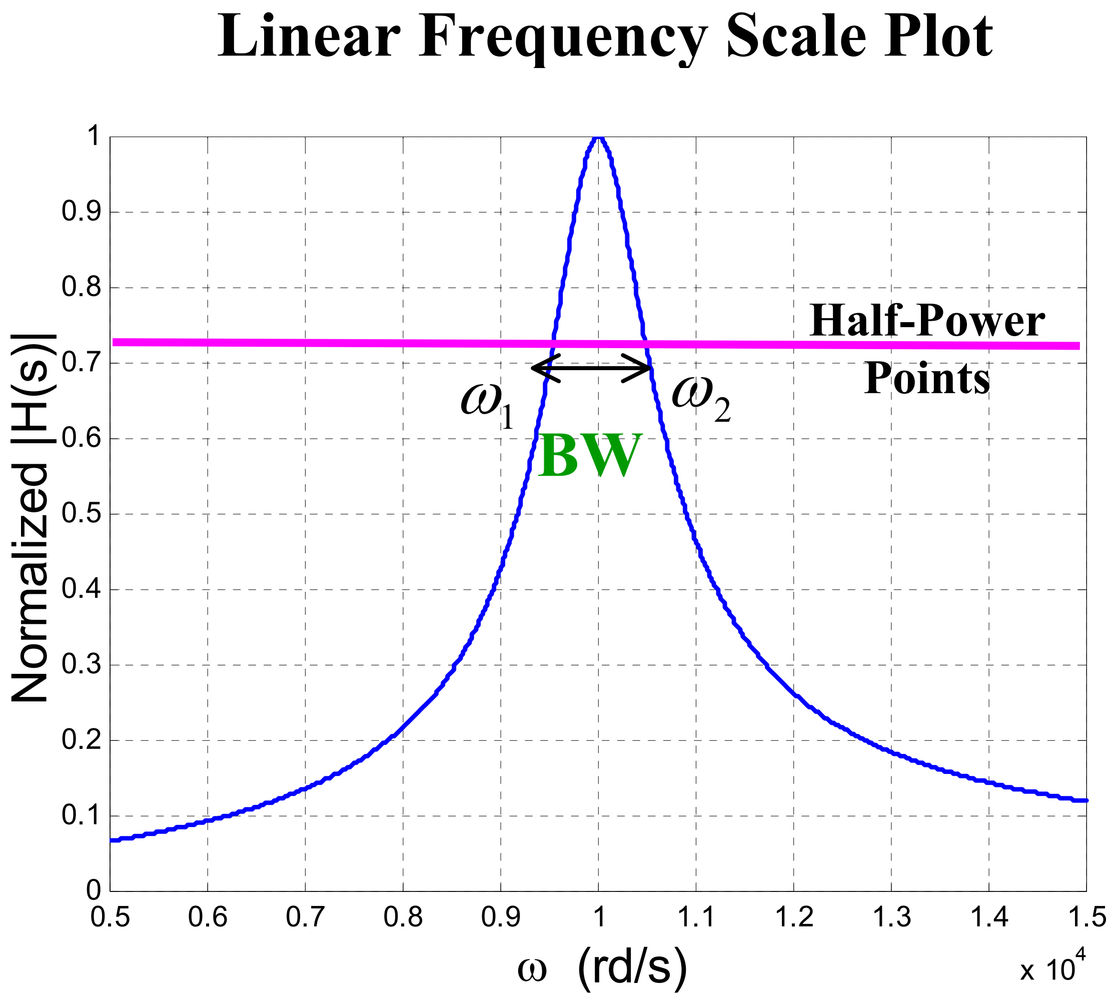
$$Q_o = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Bandwidth

$R = 2\Omega$

$C = 5\mu F$

$L = 2mH$



$$\frac{|\mathbf{H}(j\omega_r)|}{\sqrt{2}}$$

$$BW = \omega_2 - \omega_1$$

Quality Factor and Bandwidth

$$\mathbf{H}(j\omega) = \frac{\mathbf{I_o(s)}}{\mathbf{V_i(s)}} = \mathbf{Y(s)} = \frac{N(\mathbf{s})}{R + \mathbf{s}L + \frac{1}{\mathbf{s}C}} = \frac{N(j\omega)}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\mathbf{H}(j\omega_r) = \frac{N(j\omega)}{R}$$

$$Q_o = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$\frac{\mathbf{H}(j\omega)}{\mathbf{H}(j\omega_r)} = \frac{1}{1 + j\left(\omega\frac{L}{R} - \frac{1}{\omega RC}\right)} = \frac{1}{1 + jQ_o\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}$$

Quality Factor and Bandwidth

$$\frac{|\mathbf{H}(j\omega)|}{|\mathbf{H}(j\omega_r)|} = \frac{1}{\sqrt{1+Q_o^2\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$Q_o^2\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right)^2 = 1 \quad \Rightarrow \quad Q_o\left(\frac{\omega}{\omega_o}-\frac{\omega_o}{\omega}\right) = \pm 1$$

$$\omega = \pm\frac{\omega_o}{2Q_o} \pm \omega_o\sqrt{\left(\frac{1}{2Q_o}\right)^2 + 1}$$

Quality Factor and Bandwidth

Taking only positive roots:

$$\omega_1 = \omega_o \left[\frac{-1}{2Q_o} + \sqrt{\left(\frac{1}{2Q_o}\right)^2 + 1} \right]$$

$$\omega_2 = \omega_o \left[\frac{+1}{2Q_o} + \sqrt{\left(\frac{1}{2Q_o}\right)^2 + 1} \right]$$

$$BW = \omega_2 - \omega_1 = \frac{\omega_o}{Q_o}$$

Quality Factor and Bandwidth

It can also be shown that: $\omega_1 \omega_2 = (\omega_o)^2$

→ The resonant frequency is the geometric mean of the two half power frequencies.

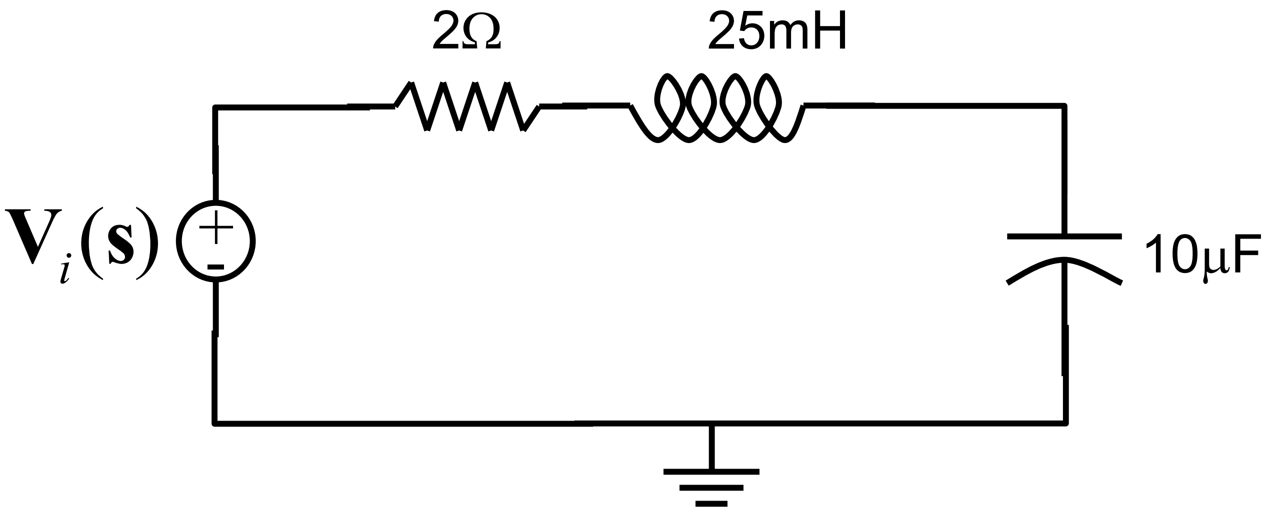
→ The bandwidth is referred to as half-power or 3dB bandwidth.

→ For a series RLC circuit, Q is inversely proportional to the bandwidth.

$$Q_o = \frac{\omega_o}{BW} = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

→ A high Q series RLC circuit has a small resistance and a small bandwidth (circuit is very *selective*).

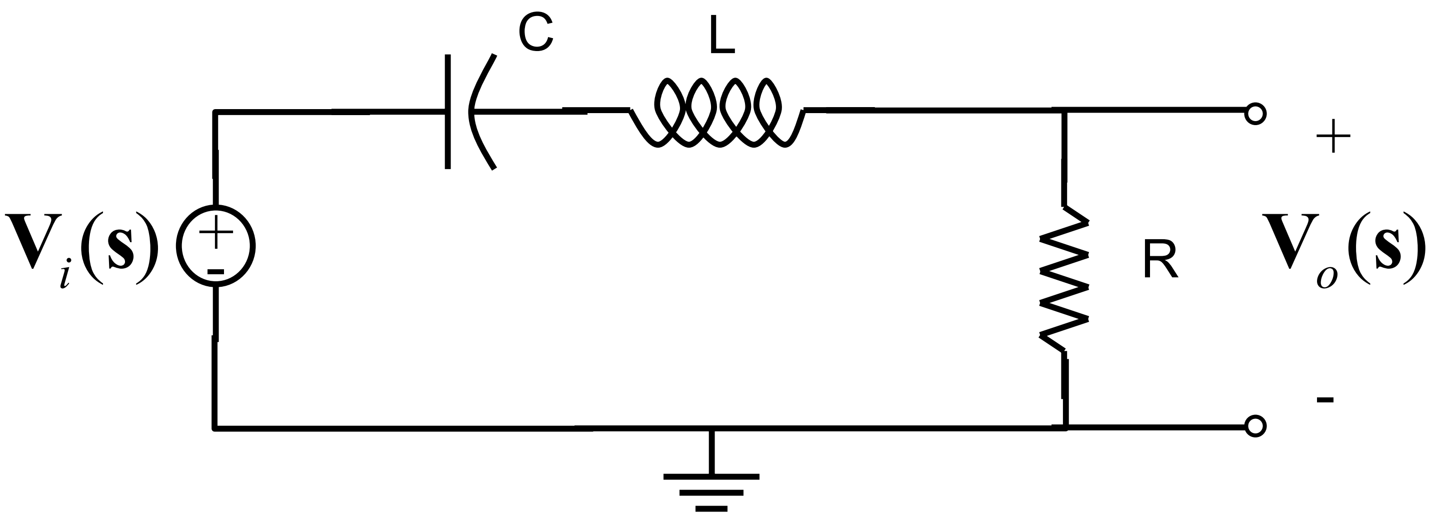
Example



Resonant frequency $\omega_o = \sqrt{\dfrac{1}{LC}} = 2000 \quad rad./sec.$

$$Q_o = \frac{\omega_o L}{R} = 25 \qquad BW = \omega_1 - \omega_2 = \frac{\omega_o}{Q_o} = 80 \quad rad./sec.$$

Example



Find R, L and C such that the above circuit operates as a ***band-pass*** filter with a center frequency of 1000 rad./sec. and bandwidth of 100 rad./sec.

Example

$$\omega_o = \sqrt{\frac{1}{LC}} = 1000 \quad rad./sec. \quad \Longrightarrow \quad LC = 10^{-6}$$

$$BW = \frac{\omega_o}{Q_o} = 100 \quad \Longrightarrow \quad Q_o = \frac{\omega_o}{100} = 10$$

$$Q_o = \frac{\omega_o L}{R} \quad \Longrightarrow \quad \frac{L}{R} = 10^{-2}$$

→ Two equations and three unknowns. Therefore we have multiple solutions.

Example

→ For example, choose:

$$C = 1\mu F$$

➡ $$L = \frac{10^{-6}}{C} = 1H$$

➡ $$R = \frac{L}{10^{-2}} = 100\Omega$$

→ The above is one set of parameters that satisfy the design requirements. We can find others.