

Transfer Functions and Frequency Response

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Unless otherwise mentioned, all problem solved here are from [1].

If the assignment consisted of examples from the textbook, see the book for the solutions.

(1) Problem 13.40.

Solution:

Let the bottom node be the ground node. Let the nodal voltage on the top node above $2s$ inductor be $\mathbf{V}_1(s)$ and the nodal voltage at the top node on the right hand side be $\mathbf{V}_2(s)$.

Applying KCL at $\mathbf{V}_1(s)$ node gives

$$\frac{\mathbf{V}_1(s) - \mathbf{V}_{s_1}(s)}{2} + \frac{\mathbf{V}_1(s)}{2s} = 0 \quad (1)$$

which can be simplified to

$$(s + 1)\mathbf{V}_1(s) = s\mathbf{V}_{s_1}(s) \quad (2)$$

Applying KCL at $\mathbf{V}_2(s)$ node gives

$$\frac{\mathbf{V}_2(s)}{1/s} + \frac{\mathbf{V}_2(s)}{2} = \mathbf{I}_{s_2}(s) \quad (3)$$

which upon simplification gives

$$(s^2 + 1)\mathbf{V}_2(s) = s\mathbf{I}_{s_2}(s) \quad (4)$$

Equations 2 and 4 can be put in matrix form:

$$\begin{bmatrix} (s + 1) & 0 \\ 0 & (s^2 + 1) \end{bmatrix} \begin{bmatrix} \mathbf{V}_1(s) \\ \mathbf{V}_2(s) \end{bmatrix} = \begin{bmatrix} s\mathbf{V}_{s_1}(s) \\ s\mathbf{I}_{s_2}(s) \end{bmatrix}$$

The poles of the system when represented in this matrix form are the roots of the determinant of the coefficient matrix on the left hand side. The determinant in this case is

$$\Delta(s) = (s + 1)(s^2 + 1)$$

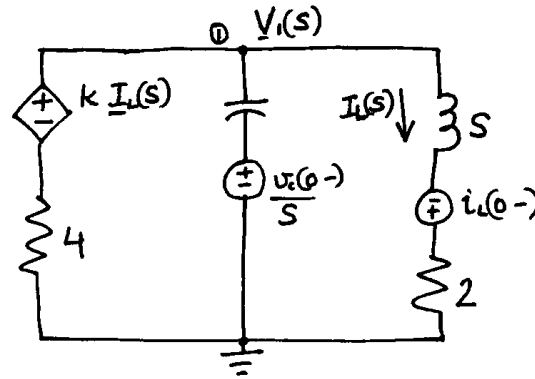
Its roots are $s_1 = -1$ and $s_{2,3} = \pm j$. Since $s_{2,3}$ do not have their real part < 0 , the system is not stable.

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(2) Problem 13.41.

Solution:

Redraw the circuit in s -domain with the initial conditions of the storage elements as show in the Figure below.



P. 13.41 circuit in s -domain

Figure 1: The given circuit in s -domain.

We will apply KCL at the node at the top to arrive at the set of equation of the form $\mathbf{M}\mathbf{v} = \mathbf{b}$ (recall the matrix form of equations in nodal analysis). Then for stability, we will see for what values of s is the determinant of \mathbf{M} zero (it gives the poles of the given circuit).

Applying KCL at the top node, we obtain

$$\frac{\mathbf{V}_1 - k\mathbf{I}_L}{4} + \frac{\mathbf{V}_1 - \frac{v_c(0-)}{s}}{2/s} + \frac{\mathbf{V}_1 + i_L(0-)}{s+2} = 0$$

We need to put \mathbf{I}_L in terms of the nodal voltages in the above equation. From the give circuit, it is given by

$$\mathbf{I}_L = k \frac{\mathbf{V}_1 + i_L(0-)}{s+2}$$

Using this value, the nodal equation can be simplified to

$$(2s^2 + 5s + 6 - k)\mathbf{V}_1 = (k - 4)i_L(0-) + 2(s + 2)v_c(0-)$$

where the coefficient of \mathbf{V}_1 is the matrix \mathbf{M} . Since in our case the circuit had only one independent node, the matrix is of size 1 by 1. The roots of the determinant of this matrix give the poles of the circuit. So, the poles of this circuit are

$$2s^2 + 5s + 6 - k = 0 \tag{5}$$

$$\implies s_{1,2} = \frac{-5 \pm \sqrt{8k - 23}}{4} \tag{6}$$

Thus for stability (real part of the poles must lie on the left hand side of the complex plane). Therefore, for stability $\sqrt{8k - 23} < 5$, or $k < 6$.

(3) Problem 13.42.

Solution:

Redraw the circuit in s -domain with the initial conditions of the storage elements as show in the Figure below.

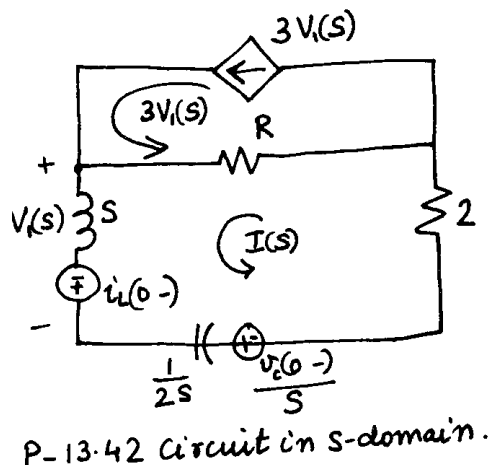


Figure 2: The given circuit in s -domain.

We will apply KVL around the two loops of the circuit. The top loop's current is already defined by the controlled current source. We will then write the mesh equation in the form of $\mathbf{M}\mathbf{i} = \mathbf{b}$ (recall the matrix form of equations in mesh analysis). Then for stability, we will see for what values of s is the determinant of \mathbf{M} zero (it gives the poles of the given circuit).

Applying KVL around the bottom loop, we obtain

$$\left(R + s + \frac{1}{2s} + 2 \right) \mathbf{I} - R3\mathbf{V}_1 = i_L(0-) - \frac{v_c(0-)}{s}$$

The voltage across the capacitor is given in terms of the mesh currents as $\mathbf{V}_1 = s\mathbf{I} - i_L(0-)$. Using this, the above equation can be simplified to

$$(2s^2(1 - 3R) + 2s(R + 2) + 1)\mathbf{I} = (1 - 3R)si_L(0-) - v_c(0-)$$

Here, the coefficient of \mathbf{I} is the M matrix. And since the circuit has only one independent loop, the matrix is of size 1 by 1. For stability, we check the roots of the determinant of that matrix

$$s_{1,2} = \frac{-2(R + 2) \pm \sqrt{4(R + 2)^2 - 8(1 - 3R)}}{4(1 - 3R)}$$

Hence, for stability

$$\begin{aligned} \sqrt{4(R + 2)^2 - 8(1 - 3R)} &< 2(R + 2) \\ -8(1 - 3R) &< 0 \\ R &< 1/3 \end{aligned} \tag{7}$$

Therefore, the circuit is unstable for $R > 1/3$.

(4) Problem 14.9.

Solution:

The given differential equation is

$$\frac{d^2v(t)}{dt^2} - v_o(t) = v_i(t)$$

which in s -domain is

$$\mathbf{V}_o(s)s^2 - \mathbf{V}_o(s) = \mathbf{V}_i(s)$$

Hence, from the above equation, the transfer function is

$$\mathbf{H}(s) = \frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} = \frac{1}{s^2 - 1}.$$

The frequency transfer function is then

$$\mathbf{H}(\omega) = \mathbf{H}(s)|_{s=j\omega} = \frac{1}{(j\omega)^2 - 1} = \frac{1}{-\omega^2 - 1}.$$

Hence, from the frequency transfer function, the output to input $v_i(t) = 4 \cos 2t$, which in phasor form is $\mathbf{V}_i = 4\angle 0^\circ$ with $\omega = 4$ rad/s., is

$$\mathbf{V}_o = \mathbf{V}_i \mathbf{H}(\omega)|_{\omega=4} = \frac{4\angle 0^\circ}{-5} = \frac{4}{5}\angle -180^\circ$$

In time domain, the output is thus

$$v_o(t) = \frac{4}{5} \cos(2t - 180^\circ).$$

(5) Problem 14.36.

Solution:

The question asks you to draw the three impedances in complex coordinates as a function of the resonant frequency of the circuit.

In general, whenever you encounter such a problem, the first thing to do is, obviously, to determine what is the resonant frequency ω_r . To do this, you need need to work out the transfer function first. Resonant frequency is the frequency where the gain of the transfer function attains it's peak. Also, there must be complex factors in the transfer function for resonance to occur. To find ω_r , obtain the expression of the gain, $|\mathbf{H}(j\omega)|$, and determine the frequency where the gain is maximum. This can be done either by differentiating the gain w.r.t. ω and equating the derivative to zero to obtain ω_r , or by writing the gain in such a way that you have $1/\omega$ terms in it. In this form, it might be possible to easily see what value of ω gives the maximum value of the gain.

After determining the resonant frequency, the solution to the problem is straight forward. The resonant frequency of a series RLC circuit is $\omega_r = \sqrt{\frac{1}{LC}}$ (I strongly recommend that you verify this by working it out yourself, it is explained in Ch. 14 [1]). For this problem, another way to obtain the resonant frequency is to note that at this frequency, the impedance of the circuit is minimum, which is possible only when the imaginary part of the impedance is zero (or when the angle of the impedance of the circuit is zero). Either of these conditions result in the expression for the resonant frequency above.

So, once we know that in this case $\omega_r = \sqrt{\frac{1}{LC}}$, and also that $Z_R = R$, $Z_L = jL\omega$ and $Z_C = -j\frac{1}{C\omega}$, the solutions is to draw the impedances show in Table 1. Note that at $\omega = \omega_r$, the reactive impedances cancel each other out.

Table 1: (prob. 14.36) Impedances as a function of ω_r

$\omega =$	$\frac{1}{10}\omega_r$	ω_r	$10\omega_r$
$Z_R =$	R	R	R
$Z_L =$	$j\frac{1}{10}\sqrt{\frac{L}{C}}$	$j\sqrt{\frac{L}{C}}$	$j10\sqrt{\frac{L}{C}}$
$Z_C =$	$-j10\sqrt{\frac{L}{C}}$	$-j\sqrt{\frac{L}{C}}$	$-j\frac{1}{10}\sqrt{\frac{L}{C}}$

(6) Problem 14.37.

Solution:

This problem is similar to the previous one (p. 14.36). In this case however, we are going to plot the admittances.

The method to obtain ω_r is the same as before. Find the transfer function (let the output be the voltage and the input be the current), obtain the gain of the transfer function and determine the resonant frequency from there (the frequency at which the gain peaks). If you work this out, you will obtain $\omega_r = \sqrt{\frac{1}{LC}}$ (this happens to be the same as that of the series RLC circuit).

Also, the admittances are $Y_R = \frac{1}{R}$, $Y_L = -j\frac{1}{L\omega}$ and $Y_C = jC\omega$. The impedances as a function of $\omega = \omega_r$ are shown in Table 2.

Table 2: (prob. 14.37) Admittances as a function of ω_r

$\omega =$	$\frac{1}{2}\omega_r$	ω_r	$2\omega_r$
$Y_R =$	$\frac{1}{R}$	$\frac{1}{R}$	$\frac{1}{R}$
$Y_L =$	$-j2\sqrt{\frac{C}{L}}$	$-j\sqrt{\frac{C}{L}}$	$-j\frac{1}{2}\sqrt{\frac{C}{L}}$
$Y_C =$	$j\frac{1}{2}\sqrt{\frac{C}{L}}$	$j\sqrt{\frac{C}{L}}$	$j2\sqrt{\frac{C}{L}}$

(7) Problem 14.38.

Solution:

This is a design problem and it entails using some relationships to come up with some practically feasible values of components to construct the circuit.

The design criteria given are the resonant frequency, ω_r , and bandwidth B , of a series RLC circuit. In general, you would need to find the expression of the resonant frequency and the bandwidth for a given circuit. In this case however, we already know that $\omega_r = \sqrt{\frac{1}{LC}}$ and that $B = \frac{R}{L}$. Therefore

$$\omega_r = \sqrt{\frac{1}{LC}} = 50 \implies \frac{1}{LC} = 2500,$$

and

$$B = \frac{R}{L} = 4 \implies R = 4L$$

We have three unknowns but only two equations. We need to make a reasonable guess for one of the circuit elements and we can then evaluate the other two elements from the above relations. Since a practical inductor has a resistance of its own, we could start with a value for L or C and come up with the two values. The final resistance to be used can then be determined by taking the resistance of the inductor into account. For $C = 1600\mu\text{F}$, using the above two relations we get $L = 2.5\text{H}$ and $R = 10\Omega$ where R is the total resistance in the circuit.

Similarly, repeat the same exercise for a parallel RLC circuit. In this case, the expression for the bandwidth is different (see the table in Ch. 14 of [1]).

(8) Problem 14.39.**Solution:**

This design problem is similar to the previous one (14.38). In this case however, we are given the quality factor and the resonant frequency. For a series circuit, quality factor is given by $Q = \frac{\omega_r}{B} = \frac{1}{R} \sqrt{\frac{L}{C}}$. Following a reasoning similar to the one given in the previous problem's solution, come up with a reasonable set of R , L and C values. Repeat for the parallel configuration.

(9) Problem 14.61.**Solution:**

This problem gives insights in how to work with approximate values from a transfer function in a resonant circuit.

Using circuit analysis methods, solve the circuit in s -domain to obtain the expression for the impedance as seen from the two terminals. You will obtain

$$\mathbf{Z}(s) = \frac{s^2 + 10.04s + 100.4}{(s + \frac{1}{50})(s + 5010)} \quad (8)$$

Note that the roots for the numerator are complex, hence there is resonance in the circuit. The natural frequency, ω_n , from the numerator is $\omega_n = \sqrt{100.4} = 10.01$ rad/s. If you were to draw the bode gain plot of this impedance transfer function, you would consider this frequency to be the approximate resonant frequency.

To obtain the exact resonant frequency, obtain the gain of the impedance frequency transfer function and equate its imaginary part to zero. The frequency transfer function is given by

$$\mathbf{Z}(j\omega) = |\mathbf{Z}(s)|_{s=j\omega} = \frac{(100.4 - \omega^2) + j10.04\omega}{(j\omega + \frac{1}{50})(j\omega + 5010)} \quad (9)$$

This expression's imaginary part, or its angle, is zero at $\omega_r = 10.02$ rad/s. This agrees pretty well with our approximate resonant frequency above.

References

- [1] D. E. Johnson, J. R. Johnson, J. L. Hilburn, and P. D. Scott, *Electric Circuit Analysis*. Upper Saddle River, New Jersey 07458: Prentice Hall, third ed., 1997.