

# Transfer Functions and Frequency Response

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Unless otherwise mentioned, all problem solved here are from [1].

If the assignment consisted of examples from the textbook, see the book for the solutions.

(1) Problem 14.01.

**Solution:**

Given  $\mathbf{V}_i = 3\angle 30^\circ$  and  $\mathbf{H}(s)$  below, we are to find  $\mathbf{V}_o$ . Since

$$\begin{aligned}\mathbf{H}(s) &= \frac{\mathbf{V}_o(s)}{\mathbf{V}_i(s)} \\ \therefore \mathbf{V}_o(j\omega) &= \mathbf{V}_o = \mathbf{H}(s)\mathbf{V}_i(s)|_{s=j\omega} \\ &= \mathbf{H}(j\omega)\mathbf{V}_i\end{aligned}\tag{1}$$

1.

$$\begin{aligned}\mathbf{H}(s) &= \frac{1}{s+1} \\ \Rightarrow \mathbf{H}(j\omega) &= \frac{1}{j\omega+1} = \frac{1}{\sqrt{\omega^2+1}}\angle -\tan^{-1}\omega \\ \therefore \mathbf{V}_o &= \left(\frac{1}{\sqrt{\omega^2+1}}\angle \tan^{-1}\omega\right)(3\angle 30^\circ) \\ &= \frac{3}{\sqrt{\omega^2+1}}\angle 30^\circ - \tan^{-1}\omega\end{aligned}\tag{2}$$

2.

$$\begin{aligned}\mathbf{H}(s) &= \frac{5s}{s+1} \\ \Rightarrow \mathbf{H}(j\omega) &= \frac{j5\omega}{j\omega+5} = \frac{5\omega}{\sqrt{\omega^2+25}}\angle 90^\circ - \tan^{-1}\frac{\omega}{5} \\ \therefore \mathbf{V}_o &= \left(\frac{5\omega}{\sqrt{\omega^2+25}}\angle 90^\circ - \tan^{-1}\frac{\omega}{5}\right)(3\angle 30^\circ) \\ &= \frac{15\omega}{\sqrt{\omega^2+25}}\angle 120^\circ - \tan^{-1}\frac{\omega}{5}\end{aligned}\tag{3}$$

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3.

$$\begin{aligned}
 \mathbf{H}(s) &= \frac{10(s-1)}{s^3 + 2s^2 + s + 2} = \frac{10(s-1)}{(s^2+1)(s+2)} \\
 \Rightarrow \mathbf{H}(j\omega) &= \frac{10(j\omega-1)}{(j\omega^2+1)(j\omega+2)} = \frac{10\sqrt{\omega^2+1}}{|1-\omega^2|\sqrt{\omega^2+4}} \angle(\tan^{-1}(-\omega) - \tan^{-1}\frac{\omega}{2} - \angle(1-\omega^2)) \\
 \therefore \mathbf{V}_o &= \left( \frac{10\sqrt{\omega^2+1}}{|1-\omega^2|\sqrt{\omega^2+4}} \angle(\tan^{-1}(-\omega) - \tan^{-1}\frac{\omega}{2} - \angle(1-\omega^2)) \right) (3\angle 30^\circ) \\
 &= \frac{30\sqrt{\omega^2+1}}{|1-\omega^2|\sqrt{\omega^2+4}} \angle(30^\circ + \tan^{-1}(-\omega) - \tan^{-1}\frac{\omega}{2} - \angle(1-\omega^2)) \quad (4)
 \end{aligned}$$

where

$$\angle(1-\omega^2) = \begin{cases} 0^\circ & \text{if } 1-\omega^2 \geq 0 \\ 180^\circ & \text{otherwise} \end{cases}$$

(2) Problem 14.10.

**Solution:**

First, represent the circuit in  $s$ -domain and assume the current source to be  $\mathbf{I}_s(s)$ . We will replace this with the actual given current source when we find the output using the transfer function.

In the  $s$ -domain circuit, applying KVL at the top node of the resistor, whose nodal voltage is  $\mathbf{V}$ , gives

$$\mathbf{I}(s) = \mathbf{I}_s(s) + \frac{r\mathbf{I}(s) - \mathbf{V}(s)}{s}.$$

Substituting  $\mathbf{I}(s) = \mathbf{V}(s)/1$  (obtained from the circuit), and simplifying, the above equation gives

$$\begin{aligned}
 (s-r+1)\mathbf{V}(s) &= s\mathbf{I}_s(s) \\
 \therefore \mathbf{H}(s) &= \frac{\mathbf{V}(s)}{\mathbf{I}_s(s)} = \frac{s}{s+1-r} \\
 \therefore \mathbf{H}(j\omega) &= \frac{j\omega}{(1-r)+j\omega} \\
 &= \frac{\omega \angle 90^\circ}{\sqrt{(1-r)^2 + \omega^2} \angle \tan^{-1}(\frac{\omega}{1-r})} \\
 &= \frac{\omega}{\sqrt{(1-r)^2 + \omega^2}} \angle 90^\circ - \tan^{-1}(\frac{\omega}{1-r}) \quad (5)
 \end{aligned}$$

Now, the phasor of the given input is  $\mathbf{I}(j\omega) = 1\angle 0$  with  $\omega = 10$  rad/s. Thus the output phasor

given this input is

$$\mathbf{V}(j10) = \mathbf{H}(j\omega)\mathbf{I}(j\omega)|_{\omega=10} \quad (6)$$

$$= \frac{\omega}{\sqrt{(1-r)^2 + \omega^2}} \angle 90 - \tan^{-1}\left(\frac{\omega}{1-r}\right) \cdot 1 \angle 0 \Big|_{\omega=10} \quad (7)$$

$$= \frac{10}{\sqrt{(1-r)^2 + 100}} \angle 90 - \tan^{-1}\left(\frac{10}{1-r}\right) \cdot 1 \angle 0 \quad (8)$$

$$\therefore v(t) = \frac{10}{\sqrt{(1-r)^2 + 100}} \cos\left(10t + 90 - \tan^{-1}\left(\frac{10}{1-r}\right)\right) \quad (9)$$

$$= \frac{10}{\sqrt{(1-r)^2 + 100}} \sin\left(10t - \tan^{-1}\left(\frac{10}{1-r}\right)\right). \quad (10)$$

For stability, the poles must lie on the left hand side of the complex frequency plane. For this to be true in the above transfer function,  $r < 1$ .

(3) Problem 14.11.

**Solution:**

Given the circuit, redrawing it in  $s$ -domain, the transfer function is found to be

$$\mathbf{H}(s) = \frac{\mathbf{I}(s)}{\mathbf{V}_s(s)} = \frac{(1+2\alpha)s}{4(1+2\alpha)s^2 + 2s + 4}. \quad (11)$$

The input, in phasor form, is  $\mathbf{V}_s = 1 \angle -90$  with  $\omega = 2$  rad/s. Given this input, the output is

$$\mathbf{I} = \mathbf{H}(j\omega)|_{\omega=2} \mathbf{V}_s \quad (12)$$

$$= j \frac{2(1+2\alpha)}{16(1+2\alpha) + j4 + 4} \cdot 1 \angle -90 \quad (13)$$

$$= \frac{2(1+2\alpha)}{(-32\alpha - 12) - j4} \quad (14)$$

$$= \frac{2(1+2\alpha)}{\sqrt{(-32\alpha - 12)^2 + 16}} \angle -\tan^{-1} \frac{16}{-32\alpha - 12}. \quad (15)$$

The system is stable as long as it's roots are on the left hand side. It is easy to show that for the poles of the given transfer function to be  $\leq 0$ , the condition is

$$\alpha \leq -\frac{1}{2}$$

(4) Problem 14.21.

**Solution:**

Redrawing the circuit in  $s$ -domain and using mesh analysis, it is easy to show that

$$\mathbf{H}(s) = \frac{\mathbf{I}(s)}{\mathbf{I}_s(s)} = 0.5 \frac{s}{s + \frac{3}{4}}$$

Hence the circuit has a single break break frequency at  $\omega = 3/4$  rad/s.

**(5)** Problem 14.26.**Solution:**

First note that the given Bode gain plot has a number of break frequencies. We will examine segments between each consecutive break frequency. Recall that a zero of multiplicity  $r$  *increases* the slope of the gain plot by  $20r$  dB/dec., and a pole of multiplicity  $r$  *decreases* the slope by  $20r$  dB/dec.

1. The first segment starts at  $\omega \approx 0$  with a slope of  $+60$  dB/dec. The positive slope indicates that it is due to a zero, and since the line is starting from 0, that zero is at the origin. The multiplicity of this zero is  $60/20 = 3$ . Therefore, we have a factor of  $s^3$  in the numerator.
2. The second segment starts from  $\omega = 2$  and the slope changes from  $+60$  to  $-80$  dB/dec. This reduction in slope indicates a pole at 2 with a multiplicity  $(60 - -80)/20 = 7$ . Hence, we have a factor of  $(s + 2)^7$  in the denominator.
3. The third segment, from  $\omega = 10$ , has a slope of 0. Since the slope has increased from the previous segment, this means there is a zero at  $\omega = 10$  of multiplicity  $(0 - -80)/20 = 4$ . Thus the transfer function has a factor of  $(s + 10)^4$  in the numerator.
4. Similarly, as indicated by the fourth segment, there is a factor of  $(s + 100)^3$  in the numerator of the transfer function.
5. The fifth segment indicates that the transfer function has a factor of  $(s + 200)^3$  in the denominator.
6. The last segment indicates that there is a factor of  $(s + 500)^2$  in the numerator.

Hence the transfer function is

$$\mathbf{H}(s) = K \frac{s^3(s + 10)^4(s + 100)^3(s + 500)^2}{(s + 2)^7(s + 200)^3}$$

where  $K$  is some constant which we do not know since the vertical axis is not labelled.

**(6)** Problem 14.33.

**Solution:**

- Based on the given transfer function, we will plot the Bode gain plot of

$$|\mathbf{H}(j\omega)|(dB) = 20 \log | - 10 | - 20 \log |j\omega - 10|$$

From the above equation, it is clear that we have a break frequency at  $\omega = 10$ .

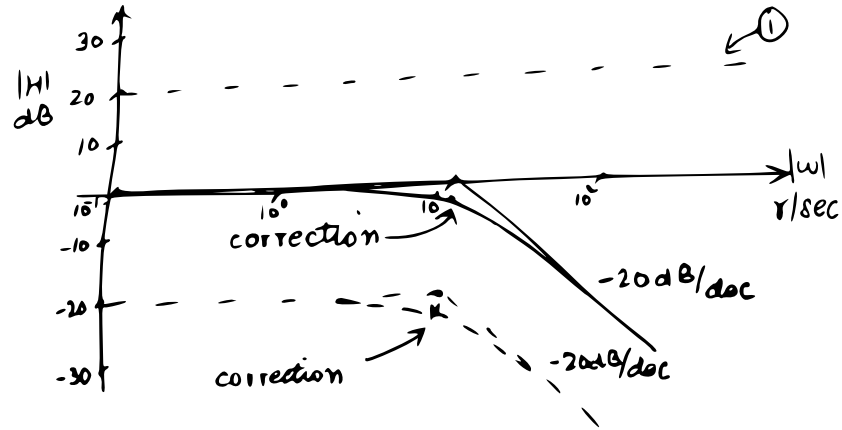


Figure 1: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curve shows the corrected gain plot.

- Based on the given transfer function, we will plot the Bode gain plot of

$$|\mathbf{H}(j\omega)|(dB) = 20 \log |3| - 20 \log |j\omega|$$

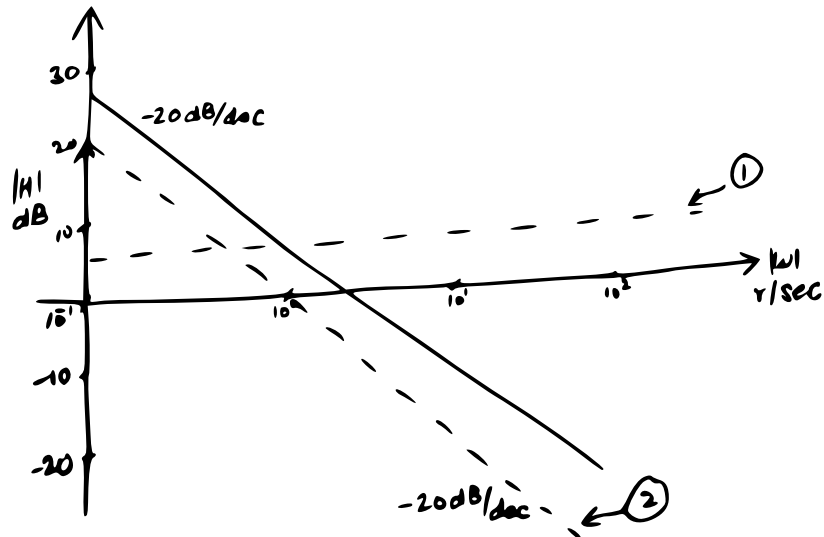


Figure 2: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curve shows the corrected gain plot.

3. Based on the given transfer function, we will plot the Bode gain plot of

$$|\mathbf{H}(j\omega)|(dB) = 20 \log |j\omega + 1| - 20 \log |j\omega + 2|$$

From the above equation, it is clear that we have a break frequency at  $\omega = 1$  and another one at  $\omega = 2$ .

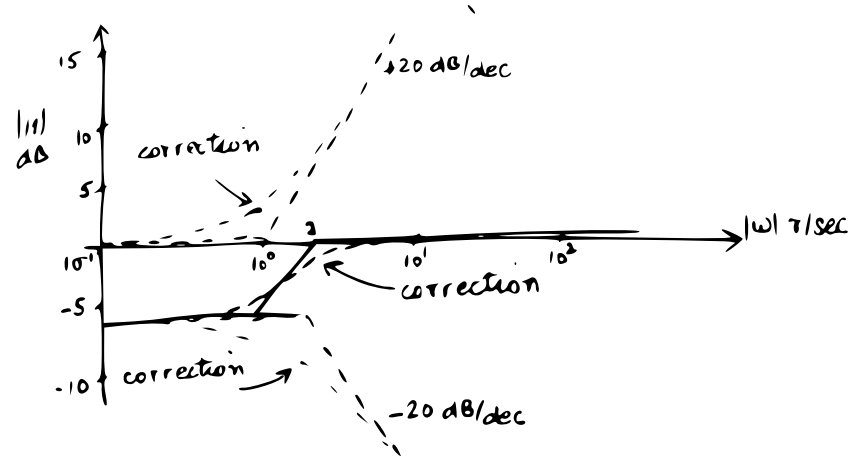


Figure 3: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curves show the corrected gain plots.

(7) Problem 14.34.

**Solution:**

1. The Bode gain plot is of the following function:

$$|\mathbf{H}(j\omega)|(dB) = -20 \log |j\omega + 1| - 20 \log |j\omega + 20|$$

The gain plot is shown below

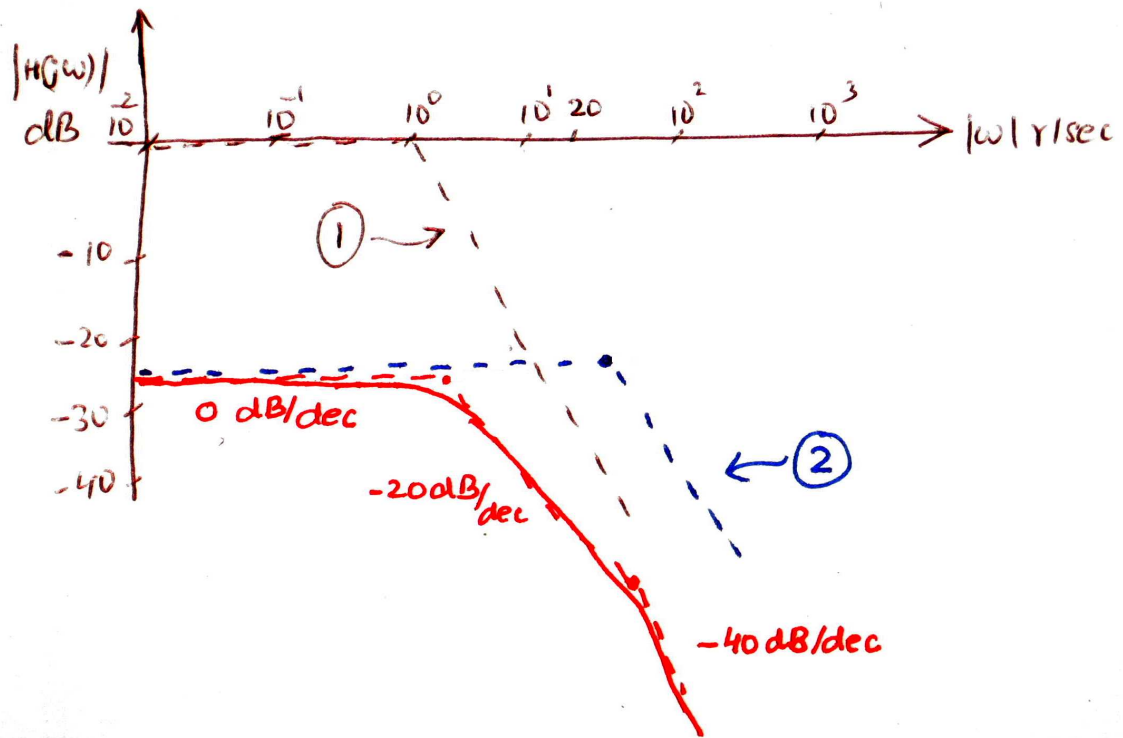


Figure 4: Bode gain plot of the given transfer function. The curve in red shows the complete transfer function; the dashed part is the approximate gain plot and the solid part is the corrected gain plot.

Based on the given transfer function, we will plot the Bode phase plot of

$$\angle H(j\omega) = \underbrace{\tan^{-1} \frac{0}{1}}_{\text{blue dashed}} - \underbrace{\tan^{-1} \frac{\omega}{1}}_2 - \underbrace{\tan^{-1} \frac{\omega}{2}}_3$$

The phase plot is given below.

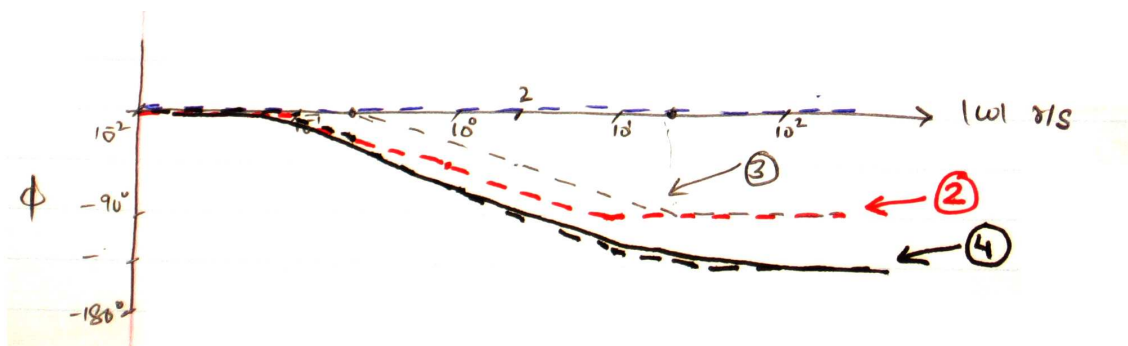


Figure 5: Bode phase plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curve shows the corrected gain plot.

2. The given transfer function be shown to be

$$|H(j\omega)|(\text{dB}) = 20 \log | -100 | - 20 \log |(10^2\omega^2) + j2 \cdot \frac{2}{20} \cdot 10\omega|$$

The corresponding gain plot is given below.

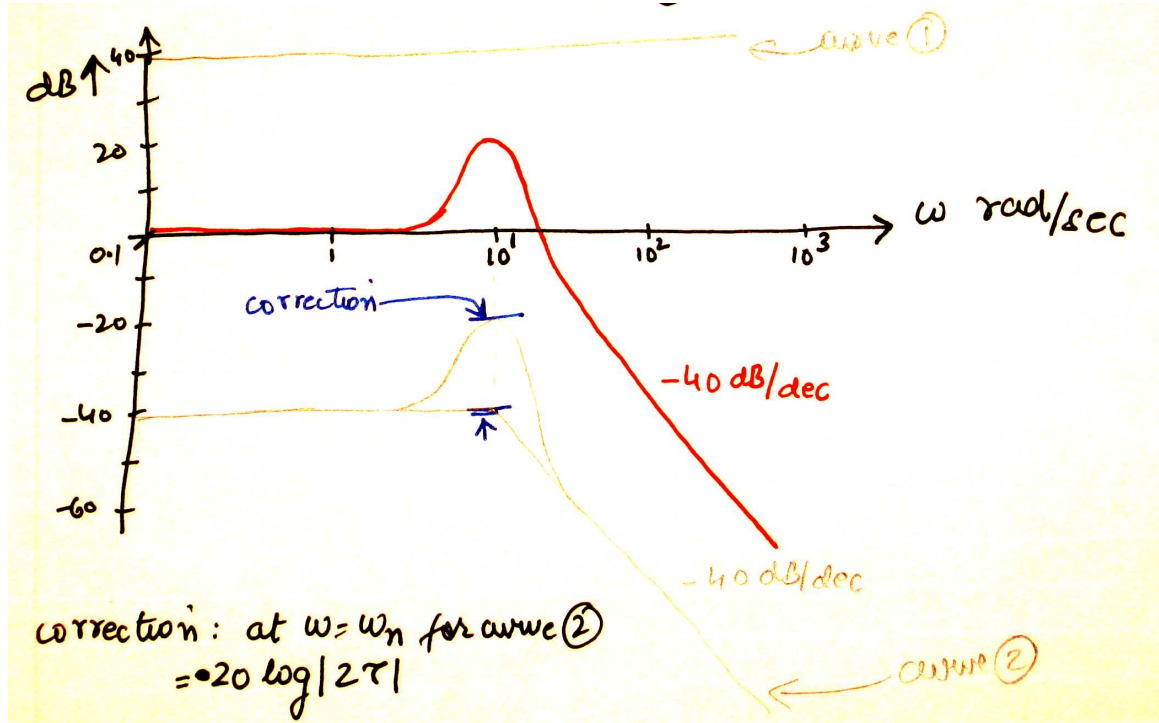


Figure 6: Bode gain plot of the given transfer function. The curve in red shows the complete transfer function.

Based on the given transfer function, we will plot the Bode phase plot of

$$\angle H(j\omega) = \underbrace{\tan^{-1} \frac{0}{-100}}_{\text{blue dashed}} - \underbrace{\tan^{-1} \frac{2\tau\omega_n\omega}{\omega_n^2 - \omega^2}}_{\text{red dashed}}$$

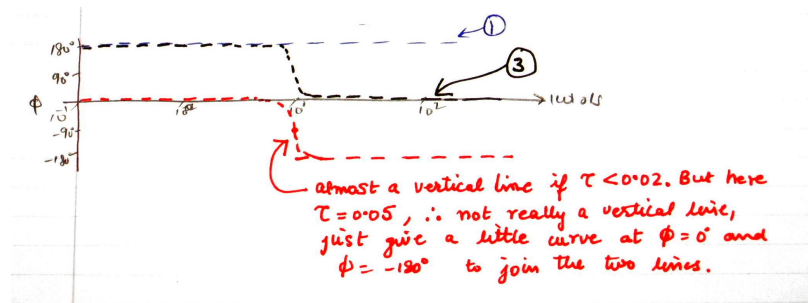


Figure 7: Bode phase plot of the given transfer function. The black dashed line shows the over all phase plot.



3. The transfer function can be simplified to

$$|\mathbf{H}(j\omega)|(\text{dB}) = 20 \log |j\omega - 41.16| + 20 \log |j\omega + 40.5| - 20 \log |j\omega + 1| - 20 \log |j\omega + 1000|$$

The corresponding gain plot is given below.

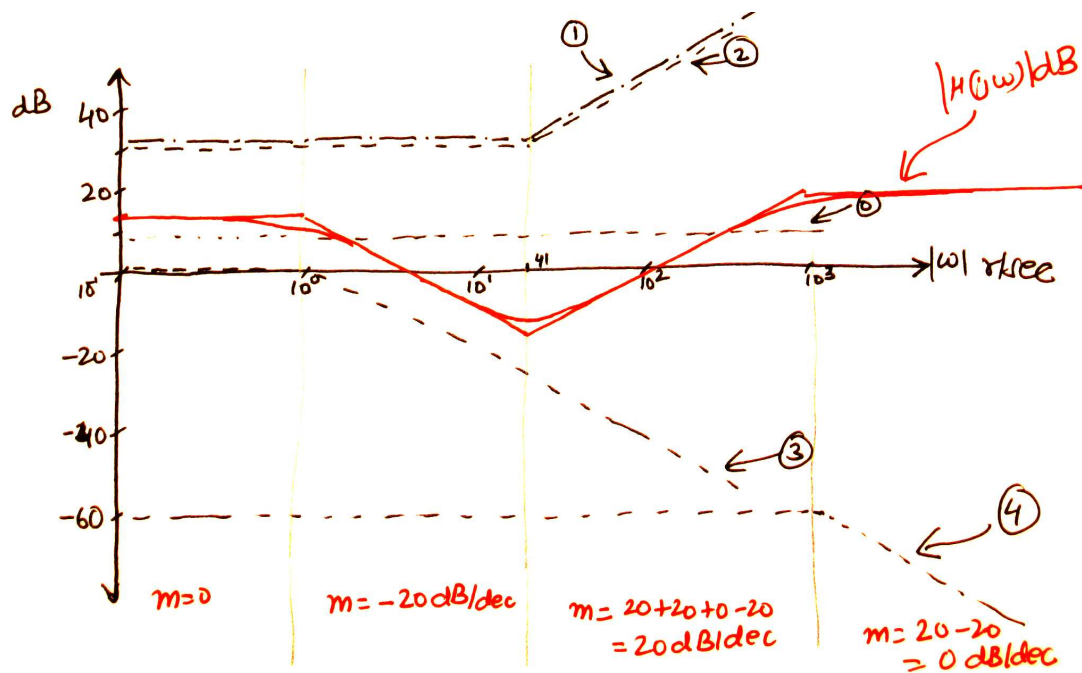


Figure 8: Bode gain plot of the given transfer function. The curve in red shows the complete transfer function.

Based on the given transfer function, the phase is given by

$$\angle \mathbf{H}(j\omega) = \underbrace{\tan^{-1} \frac{0}{-3}}_1 + \underbrace{\tan^{-1} \frac{\omega}{-41.16}}_2 + \underbrace{\tan^{-1} \frac{\omega}{40.69}}_3 - \underbrace{\tan^{-1} \frac{\omega}{1}}_{\text{red4}} - \underbrace{\tan^{-1} \frac{\omega}{1000}}_{\text{black4}}$$

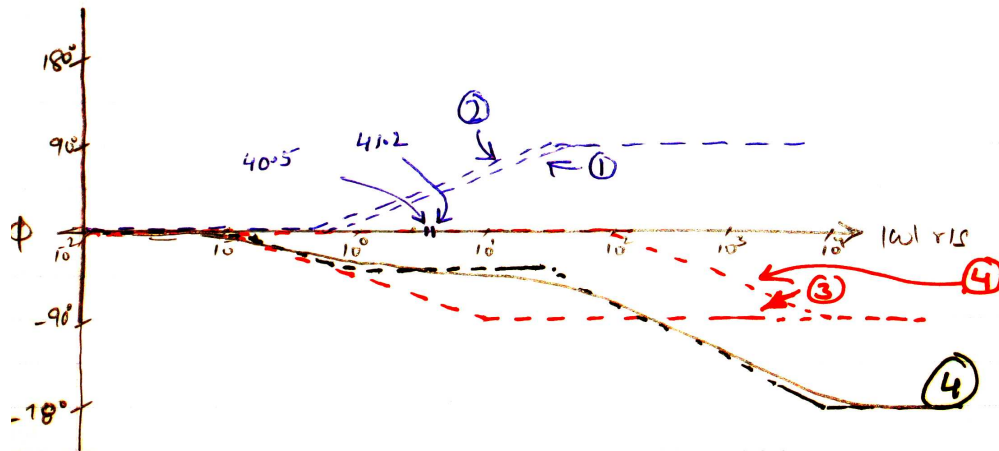
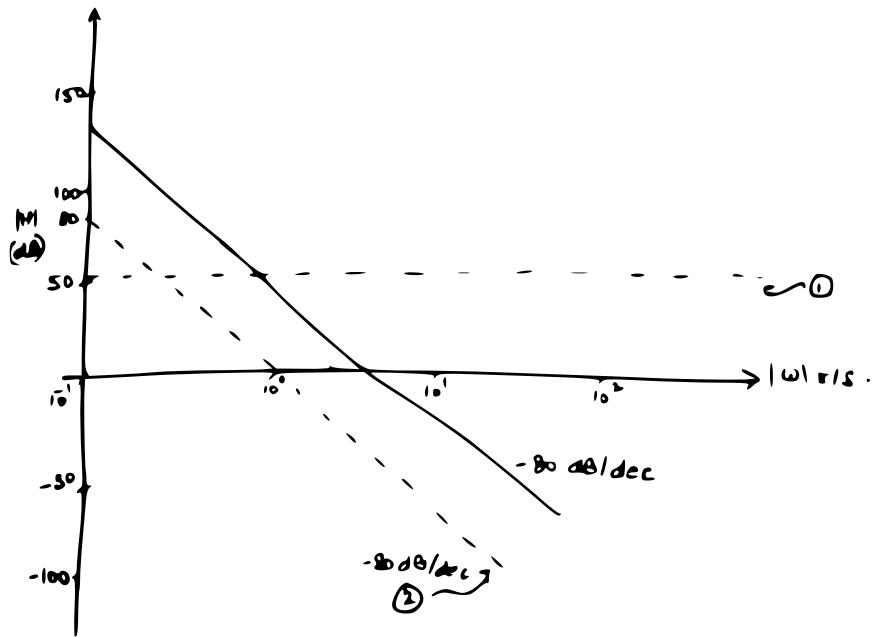


Figure 9: Bode phase plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curves show the corrected gain plots.

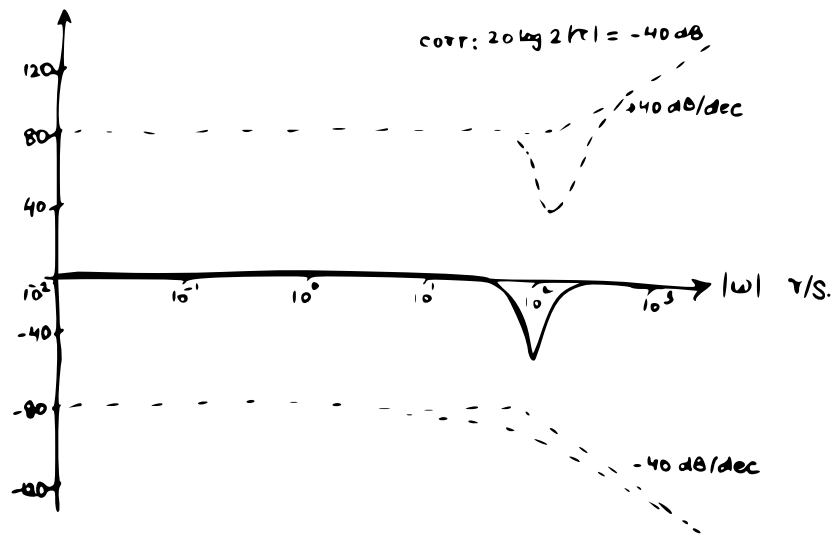
(8) Prob. 14.35.

**Solution:**



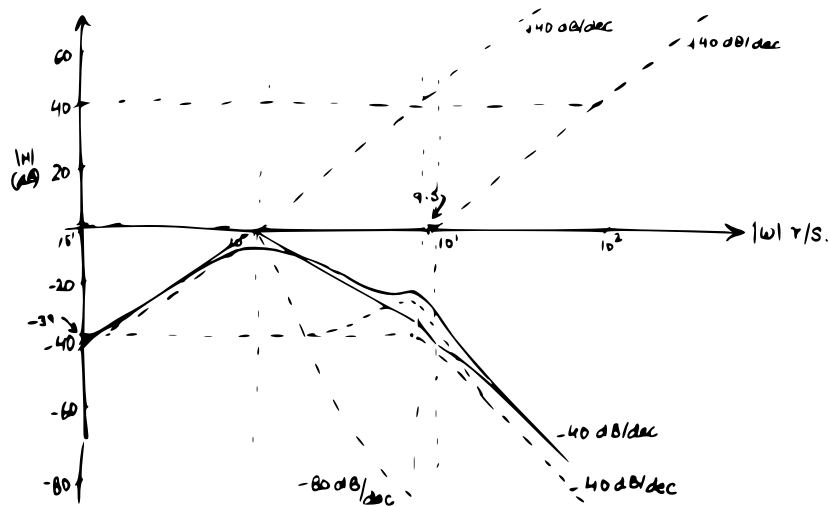
1.

Figure 10: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curves show the corrected plots.



2.

Figure 11: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curves show the corrected plots.



3.

Figure 12: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curves show the corrected plots.

(9) Problem 14.36.

**Solution:**

The question asks you to draw the three impedances in complex coordinates as a function of the resonant frequency of the circuit.

In general, whenever you encounter such a problem, the first thing to do is, obviously, to determine what is the resonant frequency  $\omega_r$ . To do this, you need need to work out the transfer function first. Resonant frequency is the frequency where the gain of the transfer function attains it's peak. Also, there must be complex factors in the transfer function for resonance to occur. To find  $\omega_r$ , obtain the expression of the gain,  $|\mathbf{H}(j\omega)|$ , and determine the frequency where the gain is maximum. This can be done either by differentiating the gain w.r.t.  $\omega$  and equating the derivative to zero to obtain  $\omega_r$ , or by writing the gain in such a way that you have  $1/\omega$  terms in it. In this form, it might be possible to easily see what value of  $\omega$  gives the maximum value of the gain.

After determining the resonant frequency, the solution to the problem is straight forward. The resonant frequency of a series  $RLC$  circuit is  $\omega_r = \sqrt{\frac{1}{LC}}$  (I strongly recommend that you verify this by working it out yourself, it is explained in Ch. 14 [1]). For this problem, another way to obtain the resonant frequency is to note that at this frequency, the impedance of the circuit is minimum, which is possible only when the imaginary part of the impedance is zero (or when the angle of

the impedance of the circuit is zero). Either of these conditions result in the expression for the resonant frequency above.

So, once we know that in this case  $\omega_r = \sqrt{\frac{1}{LC}}$ , and also that  $Z_R = R$ ,  $Z_L = jL\omega$  and  $Z_C = -j\frac{1}{C\omega}$ , the solutions is to draw the impedances show in Table 1. Note that at  $\omega = \omega_r$ , the reactive impedances cancel each other out.

Table 1: (prob. 14.36) Impedances as a function of  $\omega_r$

$\omega =$	$\frac{1}{10}\omega_r$	$\omega_r$	$10\omega_r$
$Z_R =$	$R$	$R$	$R$
$Z_L =$	$j\frac{1}{10}\sqrt{\frac{L}{C}}$	$j\sqrt{\frac{L}{C}}$	$j10\sqrt{\frac{L}{C}}$
$Z_C =$	$-j10\sqrt{\frac{L}{C}}$	$-j\sqrt{\frac{L}{C}}$	$-j\frac{1}{10}\sqrt{\frac{L}{C}}$

(10) Problem 14.37.

**Solution:**

This problem is similar to the previous one (p. 14.36). In this case however, we are going to plot the admittances.

The method to obtain  $\omega_r$  is the same as before. Find the transfer function (let the output be the voltage and the input be the current), obtain the gain of the transfer function and determine the resonant frequency from there (the frequency at which the gain peaks). If you work this out, you will obtain  $\omega_r = \sqrt{\frac{1}{LC}}$  (this happens to be the same as that of the series *RLC* circuit).

Also, the admittances are  $Y_R = \frac{1}{R}$ ,  $Y_L = -j\frac{1}{L\omega}$  and  $Y_C = jC\omega$ . The impedances as a function of  $\omega = \omega_r$  are shown in Table 2.

Table 2: (prob. 14.37) Admittances as a function of  $\omega_r$

$\omega =$	$\frac{1}{2}\omega_r$	$\omega_r$	$2\omega_r$
$Y_R =$	$\frac{1}{R}$	$\frac{1}{R}$	$\frac{1}{R}$
$Y_L =$	$-j2\sqrt{\frac{C}{L}}$	$-j\sqrt{\frac{C}{L}}$	$-j\frac{1}{2}\sqrt{\frac{C}{L}}$
$Y_C =$	$j\frac{1}{2}\sqrt{\frac{C}{L}}$	$j\sqrt{\frac{C}{L}}$	$j2\sqrt{\frac{C}{L}}$

(11) Problem 14.38.

**Solution:**

This is a design problem and it entails using some relationships to come up with some practically feasible values of components to construct the circuit.

The design criteria given are the resonant frequency,  $\omega_r$ , and bandwidth  $B$ , of a series *RLC* circuit.

In general, you would need to find the expression of the resonant frequency and the bandwidth for a given circuit. In this case however, we already know that  $\omega_r = \sqrt{\frac{1}{LC}}$  and that  $B = \frac{R}{L}$ . Therefore

$$\omega_r = \sqrt{\frac{1}{LC}} = 50 \implies \frac{1}{LC} = 2500,$$

and

$$B = \frac{R}{L} = 4 \implies R = 4L$$

We have three unknowns but only two equations. We need to make a reasonable guess for one of the circuit elements and we can then evaluate the other two elements from the above relations. Since a practical inductor has a resistance of its own, we could start with a value for  $L$  or  $C$  and come up with the two values. The final resistance to be used can then be determined by taking the resistance of the inductor into account. For  $C = 1600\mu\text{F}$ , using the above two relations we get  $L = 2.5\text{H}$  and  $R = 10\Omega$  where  $R$  is the total resistance in the circuit.

Similarly, repeat the same exercise for a parallel  $RLC$  circuit. In this case, the expression for the bandwidth is different (see the table in Ch. 14 of [1]).

(12) Problem 14.39.

**Solution:**

This design problem is similar to the previous one (14.38). In this case however, we are given the quality factor and the resonant frequency. For a series circuit, quality factor is given by  $Q = \frac{\omega_r}{B} = \frac{1}{R} \sqrt{\frac{L}{C}}$ . Following a reasoning similar to the one given in the previous problem's solution, come up with a reasonable set of  $R$ ,  $L$  and  $C$  values. Repeat for the parallel configuration.

(13) Prob. 14.44

**Solution:**

Redraw and relabel the circuit in  $s$ -domain. Assuming ideal op-amp and applying KCL at the inverting node we obtain

$$\begin{aligned} \frac{\mathbf{V}_1(s) - \mathbf{V}_0(s)}{\frac{1}{1+s}} + \frac{V_1}{1+s} &= 0 \\ \vdots & \\ \frac{\mathbf{V}_o(s)}{\mathbf{V}_1(s)} &= \frac{s^2 + 2s + 2}{(1+s)^2} \end{aligned} \tag{16}$$

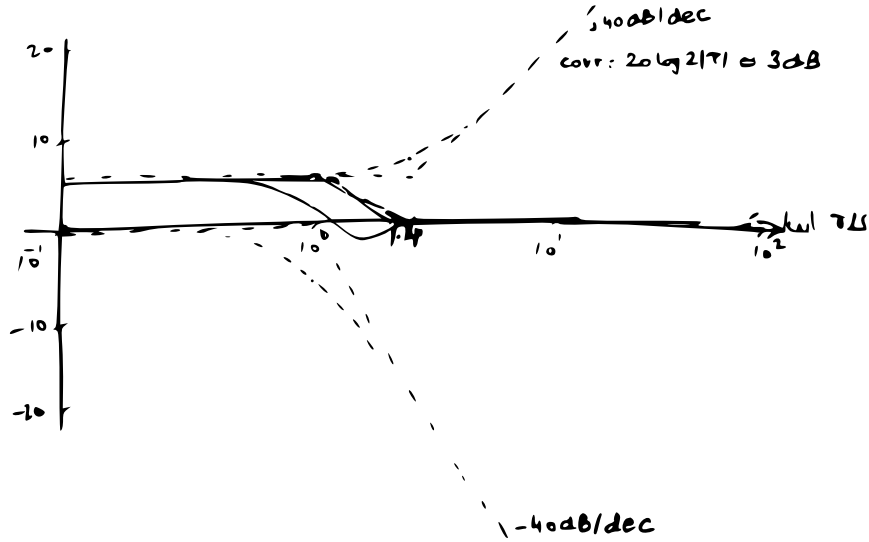


Figure 13: Bode gain plot of the given transfer function. The plot of the factors is shown with dashed lines. The solid line shows the overall gain plot. The smooth curves show the corrected plots.

## References

- [1] D. E. Johnson, J. R. Johnson, J. L. Hilburn, and P. D. Scott, *Electric Circuit Analysis*. Upper Saddle River, New Jersey 07458: Prentice Hall, third ed., 1997.