ECSE-210 Fall 2007

Loading and Two Port Networks

H. S. Sahambi*

Unless otherwise mentioned, all problem solved here are from [1].
If the assignment consisted of examples from the textbook, see the book for the solutions.

Some two port networks are defined below (see your text for other definitions and for polarities and directions of the port voltages and currents):

z-parameters

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} = \begin{bmatrix}z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix} \tag{1}$$

y-parameters

$$\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix} = \begin{bmatrix}y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} \tag{2}$$

t-parameters, or transmission line parameters

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{I}_1\end{bmatrix} = \begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_2\\ -I_2\end{bmatrix} \tag{3}$$

(1) Excercise 16.2.1.

Solution:

To find the admittance parameters, we use the definition in Eq. 2 and set certain variables to zero to find others.

- $y_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0}$. Under these conditions, the dependent current source terminals are shorted togather, and applying KVL around the input mesh gives

  $$\mathbf{V}_1 = \mathbf{I}_1(1) + 0.$$

  Hence $y_{11} = 1$ S.

- From Eq. 2 $y_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0}$. Under these conditions, the input terminals are shorted together and thus the dependent current source is of 0 A, and thus may be replaced with an open circuit. We are then left just one mesh where $\mathbf{I}_1 = -\mathbf{I}_2$. Applying KVL around that mesh gives

  $$\mathbf{V}_2 = -\mathbf{I}_1 \;\therefore y_{12} = -1.$$

*To report any errors in this document, send email to the author at hsDOTsahambiATmcgillDOTca.

- From Eq. 2 $y_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1}\Big|_{\mathbf{V}_2=0}$. These conditions are same as before where we shorted the output terminals. Applying KCL at the top node of the dependent current source and using the fact that $\mathbf{V}_1 = \mathbf{I}_1$ (from y_{11} case above), we get $\mathbf{I}_2 = 0$, hence $y_{21} = 0$. Note that this means no current is flowing through the short! Can you explain why?
- From Eq. 2 $y_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2}\Big|_{\mathbf{V}_1=0}$. In this case, the dependent current source is delivering 0 A since the input voltage is zero and thus may be replaced with an open circuit. The remaining circuit gives $\mathbf{V}_2 = \mathbf{I}_2$, hence $y_{22} = 1$ S.

Consult Table 16.2 in your textbook. Note that to convert from $y-$parameters to $t-$parameters, we need to divide the matrix by y_{21} which is zero. Hence, for the given $y-$parameters, $t-$parameters do not exist.

(2) Problem 16.01.

Solution:

In this problem, we are given the source on the left hand port and a terminator circuit on the right hand side and we are supposed to find the port variables $(\mathbf{V}_1, \mathbf{V}_2, \mathbf{I}_1, \mathbf{V}_2)$.

Applying KVL around the input port, we obtain

$$\mathbf{V}_1 = 30 - 9\mathbf{I}_1 \tag{4}$$

Applying KVL around the output loop, we obtain

$$\mathbf{V}_2 = -6\mathbf{I}_2 \tag{5}$$

Using these two voltage values (and the given Z matrix) in Eq. 1 and simplifying, we obtain

$$\begin{bmatrix} 30 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 & -8 \\ -6 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \tag{6}$$

and solving these two equations gives

$$\mathbf{I}_1 = \frac{75}{32} \tag{7}$$

$$\mathbf{I}_2 = \frac{15}{16} \tag{8}$$

Using these equations and Eqs. 4 and 5 gives

$$\mathbf{V}_1 = \frac{285}{32} \tag{9}$$

$$\mathbf{V}_2 = -\frac{45}{8} \tag{10}$$

(3) Problem 16.04.

Solution:

To find the y-parameters, we use the open circuit and short circuit tests.

- To find $y_{11} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_1}\right|_{\mathbf{V}_2=0}$, we note that it is nothing but the admittance seen by a source on the input terminals while the output port terminals are terminated in a short circuit. In this case (looking into the circuit from the input ports), note that we have two impedances in series, one at the top and one at the bottom and both joined together by the short circuit at the output port. Each of these impedances are a parallel combination of s, $2/s$ and $s/3$ ohms impedances. This series combination has a total impedance of $\frac{4s}{s^2+8}$ Ω. Hence

$$y_{11} = \frac{s^2+8}{4s} \text{ S}$$

- Similar reasoning holds for y_{22}, hence

$$y_{22} = \frac{s^2+8}{4s} \text{ S}$$

- For $y_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0}$, we short circuit the input port and apply a 1 V source at the output port. In this case, we need to find the current entering the top input terminals.

  Let the node that is obtained by shorting the input ports be a with a nodal voltage of $\mathbf{V}_a$. Applying KCL at this node while grounding the bottom right hand side node, we obtain

$$\frac{\mathbf{V}_a - 1}{\frac{2s}{s^2+8}} + \frac{\mathbf{V}_a}{s/3} + \frac{\mathbf{V}_a - 1}{s/3} + \frac{\mathbf{V}_a}{\frac{2s}{s^2+8}} = 0$$

  Simplifying this equation, we obtain

$$\mathbf{V}_a = 1/2 \text{ V}.$$

  Hence, the current entering the top left terminal of the network is (apply KCL at that node)

$$\mathbf{I}_1 = \frac{1/2 - 1}{\frac{2s}{s^2+8}} + \frac{1/2}{s/3}$$

  which upon simplification gives

$$\mathbf{I}_1 = -\frac{s^2+2}{4s}$$

  hence

$$y_{12} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{V}_1=0} = -\frac{s^2+2}{4s} \text{ S}$$

- Similar reasoning is valid for y_{21}, hence

$$y_{21} = -\frac{s^2+2}{4s} \text{ S}$$

(4) Problem 16.09.

Solution:

We are supposed to find the y-parameters of the given circuit. We can do so using the experiments to set some variables to zero and finding the others (as explained in your textbook). However, in this case, we can arrive at the parameters by applying KCL at the input and output ports. This will work because *the bottom nodes of both ports are grounded*, so the top terminals's voltage is w.r.t to the bottom terminal and is the same as the port variables $\mathbf{V}_1$ and $\mathbf{V}_2$.

Both methods should give the same answer, it is just a matter of choice which one to use in this case.

Redraw the circuit in s-domain. Let the bottom node be the ground node. Let the top nodes of input and output ports have nodal voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ respectively with their corresponding currents $\mathbf{I}_1$ and $\mathbf{I}_2$ (both entering the ports).

Applying KCL at the top of the input port, we obtain

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{2} + \frac{\mathbf{V}_1 - \mathbf{V}_1}{4/s}$$

which upon simplification gives

$$\mathbf{I}_1 = \frac{2+s}{4}\mathbf{V}_1 - \frac{s}{4}\mathbf{V}_2 \tag{11}$$

Similarly, applying KCL at the top node of the output port gives

$$\mathbf{I}_2 = \frac{\mathbf{V}_2 - \mathbf{V}_1}{4/s} + \frac{\mathbf{V}_2 + \mathbf{V}_2/2}{4}$$

which upon simplification gives

$$\mathbf{I}_2 = \frac{1-2s}{8}\mathbf{V}_1 + \frac{s+1}{4}\mathbf{V}_2 \tag{12}$$

From Eqs. 11 and 12, the y-parameters are

$$Y = \begin{bmatrix} \frac{2+s}{4} & -\frac{s}{4} \\ \frac{1-2s}{8} & \frac{s+1}{4} \end{bmatrix} \tag{13}$$

Finally, if the output port is terminated in a 1 Ω resistance, applying KVL around the output loop gives

$$\mathbf{V}_2 = -\mathbf{I}_2$$

and using this value of $\mathbf{I}_2$ in Eq. 12 and simplifying gives the desired transfer function

$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{2s-1}{2s+10} = \frac{s-1/2}{s+5}$$

(5) Problem 16.12.

Solution:

In this problem, we are supposed to find the z-parameters of the given circuit. Before proceeding, redraw the circuit in s-domain assuming zero initial conditions.

We know from the Z matrix that $z_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1}\Big|_{\mathbf{I}_2=0}$ (output port terminals are open circuited). Also note that z_{11} is nothing but the impedance seen by looking into the input terminals of the network when the output terminals are open circuited. Based on this observation

$$z_{11} = \frac{1}{s}\|(1+3) = \frac{3}{3s+1}\ \Omega$$

Also, since $z_{22} = \frac{\mathbf{V}_2}{\mathbf{I}_2}\Big|_{\mathbf{I}_1=0}$, it is the impedance seen by looking into the two port network with the input port open circuited, hence

$$z_{22} = 2 + \left(1 + \frac{1}{s}\right)\|2 = \frac{8s+4}{3s+1}\ \Omega$$

Now, $z_{12} = \frac{\mathbf{V}_1}{\mathbf{I}_2}\Big|_{\mathbf{I}_1=0}$ is found by applying a 1 V source at the output port while keeping the input port open circuited. In this case, note that $\mathbf{I}_2 = 1/z_{22}$, and using nodal analysis on the circuit under these conditions, it is found that $\mathbf{V}_1 = \frac{2}{8s+4}$, hence

$$z_{12} = \frac{2}{3s+1}\ \Omega$$

Finally, $z_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1}\Big|_{\mathbf{I}_2=0}$ is found by applying a 1 V source at the input terminals while keeping the output terminals open circuited. So, with a 1 V source across the input terminals while output terminals open circuited, using nodal analysis we obtain $\mathbf{V}_2 = 2/3$ V and $\mathbf{I}_1 = \frac{3s+1}{3}$, thus

$$z_{21} = \frac{2}{3s+1}\ \Omega$$

(6) Problem 16.18.

Solution:

To find the transmission line parameters, we use the definition in Eq. 3 and set certain variables to zero to find others.

- $t_{11} = \frac{\mathbf{V}_1}{\mathbf{V}_2}\Big|_{-I_2=0}$, this is the ratio of the input to output voltage with the output port open circuited. To find this, we can use any voltage source at the input. In this case, note that in the given circuit if $-I_2 = 0$, then $\mathbf{V}_1 = \mathbf{V}_2$, hence

$$t_{11} = 1$$

- From Eq. 3 $t_{12} = \left.\frac{\mathbf{V}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0}$. In this case, the output port is terminated in a short circuit and a voltage source is applied at the input port. Due to the short circuit, the controlled current source is zero, hence it is replaced by an open circuit. Under these conditions, if a voltage source of 1 V is applied at the input terminals, it is easy to find that $\mathbf{I}_2 = -1/4$, hence

$$t_{12} = 4\ \Omega$$

- From Eq. 3 $t_{21} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{-\mathbf{I}_2=0}$, *i.e.* the output port is open circuited and a voltage source is applied across the input port. Note that here the controlled current source is not zero (it depends on the output voltage). In this scenario, with 1 V source at the input, using KCL at the top input node gives $\mathbf{I}_1 = -2 + 1/10 = -19/10$ A. Hence

$$t_{21} = -1.9\ \mathrm{S}$$

- From Eq. 3 $t_{22} = \left.\frac{\mathbf{I}_1}{-\mathbf{I}_2}\right|_{\mathbf{V}_2=0}$, *i.e.* the output port is short circuit and a voltage source is applied to the input port terminals. Due the short circuiting of the output port, the voltage controlled current source is replaced by an open circuit. Then, it is easy to find that, with 1 V source at the input, $\mathbf{I}_1 = 7/20$ A, and $-\mathbf{I}_2 = 1/4$ A. Hence

$$t_{22} = \frac{7}{5}$$

(7) Problem 16.28.

Solution:

The inverse hybrid parameters representation of the two port network given is

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 1 & -(s+1) \\ s+1 & s^2+7s+8 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_2 \end{bmatrix} \tag{14}$$

Expanding this matrix equation gives us

$$\mathbf{I}_1 = \mathbf{V}_1 - (s+1)\mathbf{I}_2 \tag{15}$$
$$\mathbf{V}_2 = (s+1)\mathbf{V}_1 + (s^2+7s+8)\mathbf{I}_2. \tag{16}$$

Further, the 3 Ω resistor that is used to terminate the output port gives the output equation

$$\mathbf{V}_2 = -3\mathbf{I}_2. \tag{17}$$

The objective is to determine the ratio $\frac{\mathbf{I}_1}{\mathbf{V}_1}$, which gives the admittance seen from the input port. To achieve this, Eq. 17 can be used to eliminate $\mathbf{V}_2$ from Eq. 16, thus converting Eq. 16 to

$$\mathbf{I}_2 = -\frac{s+1}{s^2+13s+17}\mathbf{V}_1,$$

which is then used to eliminate $\mathbf{I}_2$ from Eq. 15 and to obtain the ratio

$$\frac{\mathbf{I}_1}{\mathbf{V}_1} = Y(s) = \frac{2s^2+15s+18}{(2s+3)(s^2+13s+17)}.$$

The above equation, which gives the desired admittance in s-domain, can be used to get the admittance at $\omega = 10$ rad/s by setting $s = j\omega$ with $\omega = 10$. This gives

$$Y(j\omega)|_{\omega=10} = \left.\frac{-2\omega^2+18+j15}{(3+j2\omega)(-\omega^2+17+j13\omega)}\right|_{\omega=10} = \cdots = 0.076\angle{-63.46^\circ}\ \text{S}.$$

References

[1] D. E. Johnson, J. R. Johnson, J. L. Hilburn, and P. D. Scott, *Electric Circuit Analysis*. Upper Saddle River, New Jersey 07458: Prentice Hall, third ed., 1997.