ECSE-210 Fall 2007

Resonance and Filters

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Unless otherwise mentioned, all problem solved here are from [1].
If the assignment consisted of examples from the textbook, see the book for the solutions.

(1) Excercise 14.4.1

Solution:

Note that the given equation is the one whose solution will provide the upper and lower cut off frequencies.

$$
\begin{aligned}
|\mathbf{Y}(j\omega)| &= \frac{1}{\sqrt{2}R} \\
\frac{1}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} &= \frac{1}{\sqrt{2}R} \\
\frac{1}{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2} &= \frac{1}{2R^2} \\
R^2 &= \left(L\omega + \frac{1}{C\omega}\right)^2 \\
\left(\omega^2 - \frac{1}{CL}\right)^2 &= \left(\frac{R\omega}{L}\right)^2
\end{aligned}
$$

(2) Excercise 14.4.2.

Solution:

Solution to the following equation gives the upper and lower cut off frequencies:

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$$\left(\omega^2 - \frac{1}{CL}\right)^2 = \left(\frac{R\omega}{L}\right)^2$$
$$\omega^2 - \frac{1}{CL} = \pm\frac{R\omega}{L}$$
$$\omega^2 - \frac{1}{CL} \pm \frac{R\omega}{L} = 0$$

The above quadratic equation has four solution, two for each of the plus and minus sings with the last term. Note that, $R > 0, C > 0, L > 0$, and only positive solutions are valid.

With the plus sign, the solutions are:

$$\omega_{1,2} = \frac{\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}}{2}$$

In the above equation, only the plus sign gives the positive solution:

$$\omega_1 = \frac{\frac{R}{L} + \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}}{2}$$

With the negative sign of the original equation, the solutions are:

$$\omega_{3,4} = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}}{2}$$

In this case also, only the plus sign gives the positive solution:

$$\omega_3 = \frac{-\frac{R}{L} + \sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}}{2}$$

Hence, the bandwidth is

$$B = \omega_1 - \omega_3 = \frac{R}{L}$$

(3) Excercise 14.4.3.

Solution:

For a series RLC circuit, we have $\omega_r = 1/\sqrt{LC}$, $B = R/L$ and $Q = R\sqrt{\frac{L}{C}}$. Given $C = 100\ \mu F$ gives $L = 1$H. Thus, $R = 5000\ \Omega$.

(4) Excercise 14.6.1.

Solution:

From the given figure

$$\mathbf{H}(s) = \frac{\mathbf{V}_L(s)}{\mathbf{V}_{in}(s)} = \frac{s}{s + R/L}$$

Thus,

$$\mathbf{H}(j\omega) = \frac{j\omega}{j\omega + R/L}$$

Thus, the magnitude of the transfer function is

$$|\mathbf{H}(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}} = \frac{1}{\sqrt{1 + \frac{R^2}{\omega^2 L^2}}}$$

If the Bode gain plot is drawn for this transfer function, it will show that the the system is a high pass filter. Moreover, from the magnitude of the transfer function it is clear that the magnitude reaches one as $\omega \to 0$, and it reaches zero as $\omega \to 0$. Thus it is a high pass filter with the maximum gain of one.

For the half power frequency, we note that power is propotional to the square of the magnituded, thus, the maximum power is the square of the maximum magnitude. Hence, if the half power is at $\omega = \omega_c$, then

$$\begin{aligned} |\mathbf{H}(j\omega_c)|^2 = \frac{1}{2}|\mathbf{H}(j\omega)|^2_{\max} &= \frac{1}{2} \\ \frac{1}{1 + \frac{R^2}{\omega_c^2 L^2}} &= \frac{1}{2} \end{aligned}$$

which gives $\omega_c = R/L$.

Similarly, filter given by

$$\mathbf{H}(s) = \frac{\mathbf{V}_R(s)}{\mathbf{V}_{in}(s)}$$

can be shown to be a low pass filter with $\omega_c = R/L$.

(5) Excercise 14.6.2.

Solution:

The given transfer function is

$$\mathbf{H}(s) = \frac{s^2}{s^2 + 2\sqrt{2}\omega_n s + \omega_n^2}$$

Dividing both sides by s^2, we obtain

$$\mathbf{H}(s) = \frac{}{1 + \frac{2\sqrt{2}\omega_n}{s} + \frac{\omega_n^2}{s^2}}$$

Thus, the frequency transfer function is

$$\begin{aligned}\mathbf{H}(j\omega) &= \frac{1}{1 + \frac{2\sqrt{2}\omega_n}{j\omega} + \frac{\omega_n^2}{(j\omega)^2}} \\ &= \frac{1}{1 - \frac{\omega_n^2}{\omega^2} - j\frac{2\sqrt{2}\omega_n}{\omega}}\end{aligned}$$

The gain of the frequency transfer function is then

$$|\mathbf{H}(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega_n^2}{\omega^2}\right)^2 + \left(\frac{2\sqrt{2}\omega_n}{\omega}\right)^2}}$$

From the above equation, it is clear that the gain decreases as ω decreases, and the gain increases as ω increases. Thus the system is a high pass filter.

This can also be shown by draswing a Bode gain plot (by assuming some value for the natural frequency ω_n).

Now, the maximum gain occurs when $\omega = 0$, which is

$$|\mathbf{H}(j\omega)|_{\max} = |\mathbf{H}(j\omega)|_{\omega=0} = 1$$

The half power frequency is thus the solution to the following equation

$$\begin{aligned}|\mathbf{H}(j\omega)|^2 &= \frac{1}{2}|\mathbf{H}(j\omega)|_{\max} = \frac{1}{2} \times 1 \\ \frac{1}{\left(1 - \frac{\omega_n^2}{\omega^2}\right)^2 + \left(\frac{2\sqrt{2}\omega_n}{\omega}\right)^2} &= \frac{1}{2} \\ &\cdots \\ \omega^4 - 6\omega_n^2\omega^2 - \omega_n^4 &= 0\end{aligned}$$

The solution to the above quadratic equation is $\omega = 2.48\omega_n$ (after discarding the negative solutions). This is thus the half power frequency of the given transfer function.

(6) Problem 14.23.

Solution:

The exact Bode gain plot for the given factor is given by

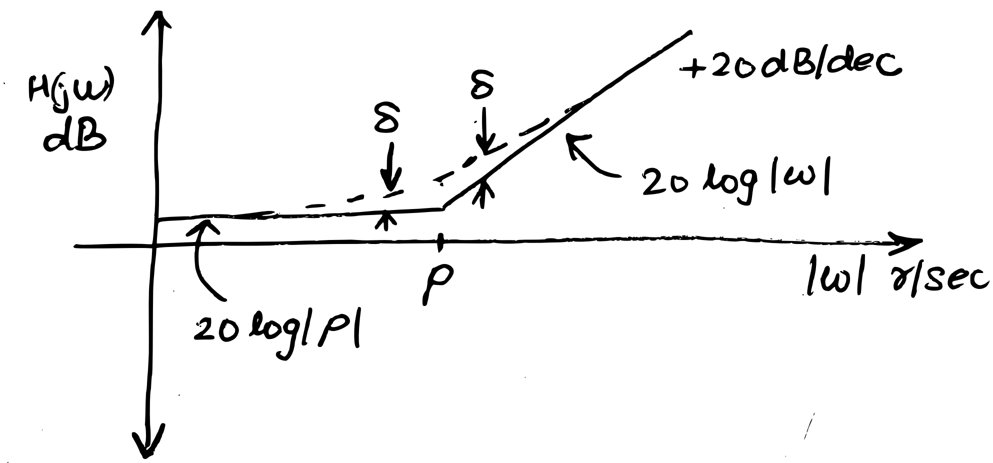
$$|\mathbf{H}j\omega|(\text{dB}) = 20\log|j\omega - p|$$

and the apporoximate Bode gain plot is given by two equations

$$|\mathbf{H}j\omega|(\text{dB}) = 20\log|p|;\ \omega << |p|$$

and

$$|\mathbf{H}j\omega|(\text{dB}) = 20\log|\omega|;\ \omega >> |p|$$



Note that there is a difference between the exact and the two approximate curves, as shown in the figure above. The difference is maximum at $\omega = |p|$ and this maximum value is 3 dB. At we move away from this break frequency in either direction, the difference gradually decreases. Thus, there are two differences of the same magnitude on either side of the break frequency. One is between the exact plot and the first approximate plot and the second is between the exact plot and the second approximate plot. Let the difference be δ at ω_1 (for the first case) and at ω_2 (for the second case).

For ω_1

$$\begin{aligned}
\delta &= 20\log|j\omega_1 - p| - 20\log|p| \\
&= 20\log\frac{|j\omega_1 - p|}{|p|} \\
&= 20\log\frac{\sqrt{\omega_1^2 - p^2}}{p} \\
&= 10\log\frac{\omega_1^2 - p^2}{p^2} \\
\cdots \\
\omega_1 &= \left(10^{\frac{\delta}{10}} - 1\right)|p|
\end{aligned}$$

Similarly, we can show for ω_2 using the second part of the approximate curve, that

$$\omega_2 = \left(10^{\frac{\delta}{10}} - 1\right)^{-1/2}|p|$$

To determine how many decades is ω away from $|p|$, this is given by $\log\frac{\omega}{|p|}$.

Using the above equations, values for ω_1 and ω_2 for the given values of δ are given in the table below.

Table 1: Prob. 14.23

δ	ω_1 (r/s)	ω_2 r/s	decades away from $\|p\|$
3/2	$0.642\|p\|$	$1.56\|p\|$	0.192
1	$0.509\|p\|$	$1.97\|p\|$	0.298
1/2	$0.349\|p\|$	$2.86\|p\|$	0.457
0.1	$0.153\|p\|$	$6.55\|p\|$	0.816

(7) Problem 14.24.

Solution:

14.24

corrected magnitude $= 20\log_{10}\left|(\omega_n^2-\omega^2)+j\omega_n\omega\right|$

uncorrected magnitude $= 20\log_{10}\omega_n^2$, $20\log_{10}\omega^2$

difference $= 20\log_{10}\left|(\omega_n^2-\omega_1^2)+j\omega_n\omega_1\right| - 20\log_{10}\omega_n^2 = 20\log_{10}\dfrac{\sqrt{\omega_n^4-\omega_n^2\omega_1^2+\omega_1^4}}{\omega_n^2}$

or $= 20\log_{10}\left|(\omega_n^2-\omega_2^2)+j\omega_n\omega_2\right| - 20\log_{10}\omega_2^2 = 20\log_{10}\dfrac{\sqrt{\omega_n^4-\omega_n^2\omega_2^2+\omega_2^4}}{\omega_2^2}$

$$\frac{\pm x}{20} = \log_{10}\frac{\sqrt{\omega_n^4-\omega_n^2\omega_1^2+\omega_1^4}}{\omega_n^2} \qquad \frac{\pm x}{20} = \log_{10}\frac{\sqrt{\omega_n^4-\omega_2^2\omega_n^2+\omega_2^4}}{\omega_2^2}$$

Let $\omega_1 = k_1\omega_n$ Let $\omega_2 = k_2\omega_n$

$$k_1^4 - k_1^2 + \left(1-10^{\pm\frac{x}{10}}\right) = 0 \qquad k_2^4\left(10^{\pm\frac{x}{10}}-1\right) + k_2^2 - 1 = 0$$

decades from $\omega_n = \log_{10}\dfrac{\omega_2}{|\omega_n|} = \log_{10}\dfrac{|\omega_n|}{\omega_1}$

x (dB)	ω_1 (rad/s)	ω_2 (rad/s)	decades from $\lvert\omega_n\rvert$
3/2	—	—	—
1	$0.843\lvert\omega_n\rvert$, $0.538\lvert\omega_n\rvert$	$1.19\lvert\omega_n\rvert$, $1.86\lvert\omega_n\rvert$	0.074, 0.269
1/2	$0.936\lvert\omega_n\rvert$, $0.352\lvert\omega_n\rvert$	$1.07\lvert\omega_n\rvert$, $2.84\lvert\omega_n\rvert$	0.029, 0.453
0.1	$0.988\lvert\omega_n\rvert$, $0.153\lvert\omega_n\rvert$	$1.01\lvert\omega_n\rvert$, $6.55\lvert\omega_n\rvert$	0.00512, 0.816

References

[1] D. E. Johnson, J. R. Johnson, J. L. Hilburn, and P. D. Scott, *Electric Circuit Analysis*. Upper Saddle River, New Jersey 07458: Prentice Hall, third ed., 1997.