Last lecture I introduced the definition of "big O" which defined an *asymptotic upper bound* on a sequence $t(n)$. The word "asymptotic" refers to the $n \geq n_0$ condition in the definition. Let's next look at a similar definition for an asymptotic *lower bound*.

Big Omega (lower bound)

We say that $t(n)$ is $\Omega(g(n))$ – "big Omega of $g(n)$" – if there exists a positive integer n_0 and a constant $c > 0$ such that

$$t(n) \;\geq\; c\, g(n)$$

for all $n > n_0$. The idea is that $t(n)$ grows at least as fast as $g(n)$ times some constant, for sufficiently large n. Here we are concerned with a lower bound. It will be used to say that an algorithm cannot run any faster than some function of n.

Big Theta

We say that $t(n)$ is $\Theta(g(n))$ if $t(n)$ is both $O(g(n))$ and $\Omega(g(n))$. Note that for this to be possible, the constants c that are used for the big O and big Ω bounds will typically be different from each other. Indeed you should be able to see that the constants will only be the same if and only if $t(n) = cg(n)$ for some c and for some sufficiently large n.

Examples

Example 1: Upper bound of a geometric series

Show that

$$t(n) = \sum_{i=0}^{n} 3^i \quad \text{is} \quad O(3^n).$$

This is not totally obvious. For example, recall (see the tutorial of Feb. 4) that $\sum_{i=1}^{n} i$ was $O(n^2)$, so you might think that $\sum_{i}^{n} 3^i$ might be bigger than $O(3^n)$. But it is not.
Proof: First, you need to recall the formula for a geometric series

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}.$$

You can prove this either by induction (do it if you're not sure), or by inspection, namely multiplying out $(1-a)\sum_{i=1}^{n} a^i$ and notice how all the terms except two cancel out (called "telescoping"). Then,

$$\sum_{i=1}^{n} 3^i = \frac{1 - 3^{n+1}}{1 - 3} = \frac{3}{2}(3^n) - \frac{1}{2}$$

and it is easy to see that the last term is $O(3^n)$. Take $c = \frac{3}{2}$ and $n_0 = 1$.

Example 2: Lower bound on Fibonacci numbers

In the tutorial on Wed. Feb. 4, I proved an upper bound on Fibonacci numbers, namely that $F(n)$ is $O(2^n)$. Let's now prove a lower bound,

$$F(n) \in \Omega((\frac{3}{2})^n).$$

We need to find an n_0 and c such that $F(n) \geq c(\frac{3}{2})^n$ for all $n \geq n_0$. Inspecting the table below, we see that if we let $c = (\frac{2}{3})^2 = \frac{4}{9}$, then we have $F(n) \geq c(\frac{3}{2})^n$ for $n = 1, 2, 3$. So, we let $n_0 = 1$. This establishes the base case for mathematical induction.

n	F(n)	g(n) = (3/2)^n	c g(n)
--	--	-------	------
0	0	1	4/9
1	1	3/2	2/3
2	1	9/4	1
3	2	27/8	3/2
:	:	:	

We next assume the induction hypothesis[1] – namely that $F(k) \geq c(\frac{3}{2})^k$ for some $k \geq n_0$. We want to show that $F(k+1) \geq c(\frac{3}{2})^{k+1}$.

$$\begin{aligned}
F(k+1) &= F(k) + F(k-1) \\
&\geq c(\frac{3}{2})^k + c(\frac{3}{2})^{k-1} \quad \text{by induction hypothesis} \\
&= c(\frac{3}{2} + 1)(\frac{3}{2})^{k-1} \\
&> c(\frac{3}{2})^2(\frac{3}{2})^{k-1}, \text{ since } \frac{5}{2} > \frac{9}{4} \\
&= c(\frac{3}{2})^{k+1}
\end{aligned}$$

This completes the proof that $F(n)$ is $\Omega((\frac{3}{2})^n)$.

How to negate the definition of $O(\ ), \Omega(\ )$?

Sometimes one would like to prove that a function $g(n)$ is *not* an asymptotic lower or upper bound of $t(n)$. How would you do that? We want to show that the definition of $O(\ )$ or $\Omega(\ )$ does *not hold*.

Saying "$t(n)$ is *not* $O(g(n))$" means that there is *no pair* of constants c and n_0 that satisfies the definition. This is equivalent to saying that, *for any* constants $c > 0$ and $n_0 \geq 0$, there is some n with $n > n_0$ and $f(n) > cg(n)$.

Similarly, saying "$t(n)$ is *not* $\Omega(g(n))$" means that there is *no pair* of constants c and n_0 that satisfies the definition. That is, *for any* constants $c > 0$ and $n_0 \geq 0$, there is some n with $n > n_0$ and $f(n) > cg(n)$.

[1] If you are unfamiliar with mathematical induction, see the tutorial from Feb. 4 and/or see me if you are unclear.

Example: Show $3n^2 + 5n + 2$ is *not* $O(n)$.

To shown it, we take any two constants $c > 0$ and $n_0 \geq 0$. We want to show that there exists an n such that

$$3n^2 + 5n + 2 > c\,n\ .$$

Dividing both sides by n, we get $3n + 5 + \frac{2}{n} > c$. The left side gets bigger and bigger without bound as n gets bigger, whereas the right side is constant. In particular, when $n > \frac{c}{3}$, the left side is greater than the right side, which is what we wanted to show. □

Example: $n!$ **is** *not* $O(2^n)$ [**Modified Feb. 10**]

Take any two constants $c > 0$ and $n_0 > 0$ and suppose c is a positive integer.[2] We want to show that there exists an $n > n_0$ such that $n! > c2^n$.

To motivate the choice of n, compare the two products below, which each have $n - 1$ terms:

$$n! = n(n-1)(n-2)\dots 2$$

$$2^{n-1} = 2 \cdot 2 \cdot 2 \cdot \dots 2$$

It is obvious that that $n! > 2^{n-1}$ for all $n > 2$, since there is a term-by-term strict inequality for all but the first term. We can make this inequality tighter by observing that, when $n \geq 5$,

$$n!\ >\ 4 \cdot 2^{n-1}\ . \qquad (*)$$

The reason is that

$$5 \cdot 4 \cdot 3 \cdot 2\ >\ 4 \cdot (2 \cdot 2 \cdot 2 \cdot 2)$$

i.e. $120 > 64$ and there is a strict inequality for all greater terms, i.e.

$$n \cdot (n-1) \dots 7 \cdot 6\ >\ 2 \cdot 2 \dots 2.$$

We rewrite $(*)$ above as

$$n! > 2^{n+1}$$

from which it follows that

$$(n-1)! > 2^n. \qquad (**)$$

Finally, let $n = max(c, 4, n_0)$. Then

$$n! \geq c \cdot (n-1)! > c \cdot 2^n$$

where the right inequality follows from (**). But this is what we wanted to prove.

[2] Convince yourself that restricting c to be an integer doesn't affect the definition of $O(\ )$ and $\Omega(\ )$.