1 Recurrences

1. Solve the recurrence, for positive constant c,

$$T(n) = T(\frac{n}{2}) + c \cdot \log n$$

ANSWER: Let $n = 2^k$

$$T(n) = T(2^{k-1}) + ck$$
$$T(n) = T(2^{k-2}) + c(k-1) + ck$$

Repeating this, gives..

$$T(n) = T(2^{k-k}) + c(k-k) + c + 2c + 3c + \ldots kc$$

$$T(n) = 1 + c \cdot \frac{k(k+1)}{2} = 1 + c \cdot \frac{\log n(\log n + 1)}{2} \in \mathcal{O}(\log^2 n)$$

2. Solve the recurrence , where $T(1) = 1$,

$$T(n) = 2T(n-1) + 2$$

ANSWER:

$$T(n) = 2\left(2T(n-2) + 2\right) + 2 = 2^2 T(n-2) + 2^2 + 2^1$$
$$T(n) = 2^2\left(2T(n-3) + 2\right) + 2^2 + 2^1 = 2^3 T(n-3) + 2^3 + 2^2 + 2^1$$
$$T(n) = 2^k\left(2T(n-k+1)\right) + \sum_{i=1}^{k} 2^i$$

Set $k = n$, giving

$$T(n) = 2^n T(n-n+1) + \sum_{i=1}^{n} 2^i = 2^n + 2^{n-1} - 2 = \frac{3}{2}2^n - 2 \in \mathcal{O}(2^n)$$

3. Solve the recurrence, with $T(1) = 1$

$$T(n) = 5T(\frac{n}{5}) + 2n$$

ANSWER: Assume that N is a power of five.

$$\begin{aligned}
T(N) &= 5T(\frac{N}{5}) + 2N \\
&= 5^2T(\frac{N}{5^2}) + 2N + 2N \\
\ldots & \\
&= 5^iT(\frac{N}{5^i}) + \underbrace{2N + 2N + \ldots + 2N}_{i\times} \\
\ldots & \\
&= 5^{\log_5 N}T(1) + \underbrace{2N + 2N + \ldots + 2N}_{\log_5 N\times} \\
&= N + 2N\log_5 N \\
&= \Theta(N\log_5 N) \\
&= \Theta(N\log N)
\end{aligned}$$

2 Induction

1. By induction (that means, don't use a formula you already know in the proof), show that

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

ANSWER:

$$\sum_{i=1}^{1} 2i - 1 = 1 = 1^2$$

So true for $n = 1$.
Assume true for $n = k$.

$$\sum_{i=1}^{k} 2i - 1 = k^2$$

Add $2k + 1$ to both sides.

$$\sum_{i=1}^{k+1} 2k - 1 = k^2 + 2k + 1 = (k+1)^2$$

As desired.

Of course, you could also do this as

$$\sum_{i=1}^{n} 2i - 1 = 2\left(\sum_{i=1}^{n} i\right) - n = 2\frac{n(n+1)}{2} - n = n^2$$

but I asked you to do it by induction...

2. In class, you encountered the Fibonacci Sequence, defined as

$$F_n = F_{n-1} + F_{n-1}$$

with initial values $F_1 = 1$ and $F_2 = 1$.

You saw several ways to calculuate this, including recursively, and with matrices.

Another way is by calculating the following exact expression for the Fibonacci number

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Prove it by induction.

HINT: You will need to prove two base cases, F_1 and F_2, as the definition of F_n goes "back" two in the sequence.

HINT: It may be useful to let $\phi = \frac{1}{2}(1+\sqrt{5})$ and $\tau = \frac{1}{2}(1-\sqrt{5})$, and observe that both ϕ and τ satisfy the equation $x^2 = x + 1$.
This would re-write F_n as

$$\frac{1}{\sqrt{5}}\left(\phi^n - \tau^n\right)$$

, and make your algebra much simpler. You can then use the relation $x^2 = x + 1$ to simplify the expression.

ENCOURAGEMENT: I really recommend trying this WITHOUT looking at the solution(which I provided in full). It will likely take some messing around, and may be annoying. But it will be good for you!

ANSWER: We shall proceed with a strong inductive argument and make use of the definition of F_n. For ease of notation, let $\phi = \left(\frac{1+\sqrt{5}}{2}\right)$ and $\tau = \left(\frac{1-\sqrt{5}}{2}\right)$.

Case $n = 1$:

$$F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1$$

$$F_1 = 1 \Rightarrow TRUE$$

Therefore true for 1.

Case $n = 2$:

$$F_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+2\sqrt{5}+5}{4}\right) - \left(\frac{1-2\sqrt{5}+5}{4}\right)\right]$$

$$F_2 = \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] = 1 \Rightarrow F_2 = 1 \Rightarrow TRUE$$

Therefore true for 2.

We now make the induction assumption that the statement holds for i such that $1 \leq i \leq k$. This gives us, in particular, these two statements:

$$F_k = \frac{1}{\sqrt{5}}\left[\phi^k - \tau^k\right], \quad F_{k-1} = \frac{1}{\sqrt{5}}\left[\phi^{k-1} - \tau^{k-1}\right]$$

We now begin with F_{k+1}:

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \quad \textit{by definition} \\ F_{k+1} &= \frac{1}{\sqrt{5}}\left[\phi^k - \tau^k\right] + \frac{1}{\sqrt{5}}\left[\phi^{k-1} - \tau^{k-1}\right] \qquad \textit{by above} \\ F_{k+1} &= \frac{1}{\sqrt{5}}\left[(\phi^k + \phi^{k-1}) - (\tau^k + \tau^{k-1})\right] \\ F_{k+1} &= \frac{1}{\sqrt{5}}\left[\phi^{k-1}(\phi+1) - \tau^{k-1}(\tau+1)\right] \end{aligned}$$

It should be noted that ϕ and τ are the roots of the equation $x^2 = x+1$. The property that $\phi^2 = \phi+1$ and $\tau^2 = \tau + 1$ will be used here.

$$F_{k+1} = \frac{1}{\sqrt{5}}\left[\phi^{k-1}\phi^2 - \tau^{k-1}\tau^2\right] = \frac{1}{\sqrt{5}}\left[\phi^{k+1} - \tau^{k+1}\right]$$

Therefore true for $k+1$, therefore true for all positive integers.

This is a specific example of what is called a Binet recursive form. The general form applies to recursions that depend upon only the two previous elements of the sequence, with possible coefficients. Details here: http://mathworld.wolfram.com/BinetForms.html.

The Fibonacci numbers, and the related Lucas numbers, can be used to test the primality of a certain class of prime numbers: http://mathworld.wolfram.com/Lucas-LehmerTest.html
The Great Internet Mersenne Prime Search: http://www.mersenne.org/