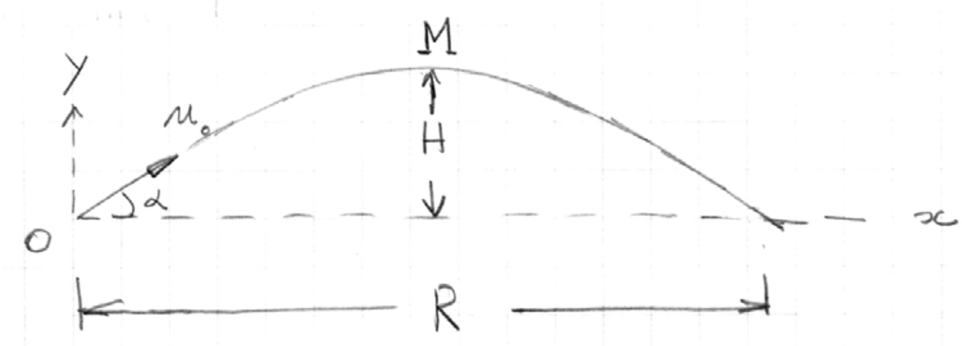
EXAMPLE: A projectile is projected with an initial velocily u_o and an inclination angle α as shown in the figure. Neglecting air friction, find (a) the expressions for the height, $H = H(\alpha)$, and the range, $R = R(\alpha)$, and (b) the radius of curvature at point O and point M of the trajectory. (c) Show that R is maximum if $\alpha = 45^o$.



Acceleration $\mathbf{a} = -g\mathbf{j}$

Velocity $\mathbf{v} = \int \mathbf{a}dt = -gt\mathbf{j} + \mathbf{C}$
The constant of integration $\mathbf{C}$ is a vector equal to the initial velocity at point O:

$$\mathbf{v} = -gt\mathbf{j} + (u_o \cos\alpha)\mathbf{i} + (u_o \sin\alpha)\mathbf{j}$$

Trajectory of the projectile is determined by the position vector

$$\mathbf{r} = \int \mathbf{v}dt = (u_o t\cos\alpha)\mathbf{i} + (u_o t\sin\alpha - \frac{1}{2}gt^2)\mathbf{j}$$

The constant of integration is zero because $\mathbf{r}(0) = 0$.

Maximum Height is reached when $v_y = u_o \sin\alpha - gt = 0$, that is, when

$$t_M = \frac{u_o \sin\alpha}{g}.$$

At this time,

$$y_M = H = u_o(\frac{u_o \sin\alpha}{g})\cos\alpha - \frac{1}{2}g(\frac{u_o \sin\alpha}{g})^2 = \frac{u_o^2 \sin^2\alpha}{2g}$$

Range: The projectile is back at the ground level when $y = u_o t \sin\alpha - \frac{1}{2}gt^2 = 0$, that is, when

$$t_R = \frac{2u_o \sin\alpha}{g}.$$

At this time,

$$x = R = u_o(\frac{2u_o \sin\alpha}{g}) \sin\alpha = \frac{u_o^2 \sin 2\alpha}{g}$$

<u>Maximum of R</u> occurs when

$$\frac{dR}{d\alpha} = \frac{2u_o^2 \cos 2\alpha}{g} = 0$$

that is

$$\alpha = 45^o \quad \text{and} \quad R_{\max} = \frac{u_o^2}{g}.$$

<u>Radius of Curvature</u> is obtained using the formula for the normal component of the acceleration:

$$a_n = \frac{v^2}{\rho}$$

At point M, $a_n = g, v = u_o \cos\alpha$. Hence,

$$\rho_M = \frac{v^2}{a_n} = \frac{u_o^2 \cos^2\alpha}{g}$$

At point O, $a_n = g\cos\alpha, v = u_o$. Hence

$$\rho_O = \frac{v^2}{a_n} = \frac{u_o^2}{g\cos\alpha}$$

In general,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(u_o \cos\alpha)^2 + (u_o \sin\alpha - gt)^2}$$

$$\hat{\tau} = \frac{\mathbf{v}}{v} = (\frac{u_o \cos\alpha}{v})\hat{\mathbf{i}} + (\frac{u_o \sin\alpha - gt}{v})\hat{\mathbf{j}}; \quad \tau_x = (\frac{u_o \cos\alpha}{v}), \tau_y = (\frac{u_o \sin\alpha - gt}{v})$$

$$a_t = \mathbf{a} \cdot \hat{\tau} = (-g\mathbf{j}) \cdot (\tau_x \mathbf{i} + \tau_y \mathbf{j}) = -g\tau_y = \frac{-u_o g \sin\alpha + g^2 t}{v}$$

$$a_n = \sqrt{|\mathbf{a}|^2 - a_t^2} = \sqrt{g^2 - a_t^2}$$

$$\rho = \frac{v^2}{a_n}$$