ANALYTICAL MECHANICS, CIVE 281
Solution to Assignment No. 1

1. The motion of a particle is defined by the position vector

$$\mathbf{r} = A(\cos t + t\sin t)\,\mathbf{i} + A(\sin t - t\cos t)\,\mathbf{j},$$

where t is expressed in seconds. Determine the values of t for which the position vector and the acceleration vector (a) perpendicular, (b) parallel. (B. & J. 11.93)

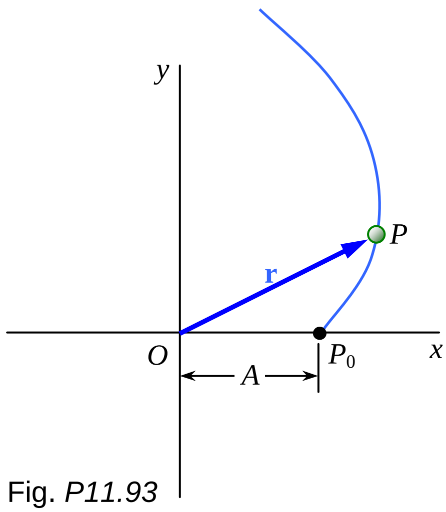


Fig. *P11.93*

Solution:

Position vector, $\mathbf{r} = A(\cos t + t\sin t)\,\mathbf{i} + A(\sin t - t\cos t)\,\mathbf{j}$

Velocity vector, $\dfrac{d\mathbf{r}}{dt} = \mathbf{v} = A(-\sin t + t\cos t + \sin t)\,\mathbf{i} + A(\cos t + t\sin t - \cos t)\,\mathbf{j}$

$$= A(t\cos t)\,\mathbf{i} + A(t\sin t)\,\mathbf{j}$$

Acceleration vector, $\dfrac{d\mathbf{v}}{dt} = \mathbf{a} = A(-t\sin t + \cos t)\,\mathbf{i} + A(t\cos t + \sin t)\,\mathbf{j}$

(a) For the position vector, **r** and acceleration vector, **a** to be perpendicular,

$$\mathbf{r} \cdot \mathbf{a} = 0$$

Definition of dot product of two vectors
$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$

$$[A(\cos t + t\sin t)\,\mathbf{i} + A(\sin t - t\cos t)\,\mathbf{j}] \cdot [A(-t\sin t + \cos t)\,\mathbf{i} + A(t\cos t + \sin t)\,\mathbf{j}] = 0$$

$$A^2\left[(\cos t + t\sin t)(-t\sin t + \cos t) + (\sin t - t\cos t)(t\cos t + \sin t)\right] = 0$$

$$(\cos t + t\sin t)(-t\sin t + \cos t) + (\sin t - t\cos t)(t\cos t + \sin t) = 0$$

$$\left(-t\sin t\cos t - t^2\sin^2 t + \cos^2 t + t\sin t\cos t\right) + \left(t\sin t\cos t - t^2\cos^2 t + \sin^2 t - t\sin t\cos t\right) = 0$$

$$-t^2\sin^2 t + \cos^2 t - t^2\cos^2 t + \sin^2 t = 0$$

$$\left(1 - t^2\right)\sin^2 t + \left(1 - t^2\right)\cos^2 t = 0$$

$$\left(1 - t^2\right)\left(\sin^2 t + \cos^2 t\right) = 0$$

Trigonometric pythagorean identity
$\sin^2 t + \cos^2 t = 1$

Therefore, $1 - t^2 = 0$

$t^2 = 1$

***t* = 1 s**

(b) For the position vector, **r** and acceleration vector, **a** to be perpendicular,

$$\mathbf{r} \times \mathbf{a} = 0$$

Definition of cross product of two vectors

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\mathbf{i}}\left(A_y B_z - A_z B_y\right) + \hat{\mathbf{j}}\left(A_z B_x - A_x B_z\right) + \hat{\mathbf{k}}\left(A_x B_y - A_y B_x\right)$$

$$\left[A(\cos t + t\sin t)\,\mathbf{i} + A(\sin t - t\cos t)\,\mathbf{j}\right] \times \left[A(-t\sin t + \cos t)\,\mathbf{i} + A(t\cos t + \sin t)\,\mathbf{j}\right] = 0$$

$$\left[A^2(\cos t + t\sin t)(t\cos t + \sin t) - A^2(\sin t - t\cos t)(-t\sin t + \cos t)\right]\hat{\mathbf{k}} = 0$$

$$A^2\left[t\cos^2 t + t^2\sin t\cos t + \sin t\cos t + t\sin^2 t - \left(-t\sin^2 t + t^2\sin t\cos t + \sin t\cos t - t\cos^2 t\right)\right]\hat{\mathbf{k}} = 0$$

$$\left(2t\sin^2 t + 2t\cos^2 t\right)\hat{\mathbf{k}} = 0$$

$$2t\left(\sin^2 t + \cos^2 t\right)\hat{\mathbf{k}} = 0$$

$$2t = 0$$

***t* = 0**

2. At a general time t, a particle has an acceleration

$$\mathbf{a} = 4e^{-2t}\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 6\cos(3t)\hat{\mathbf{k}}$$

If this particle starts from the origin at $t = 0$ with a velocity of $2\hat{\mathbf{j}}$, find its velocity and position at a general time t.

Solution:

Acceleration vector, $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 4e^{-2t}\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 6\cos(3t)\hat{\mathbf{k}}$

Since the acceleration is a given function of t, integration of the above function corresponding to the initial conditions $t = 0$ and $\mathbf{v} = \mathbf{v}_0$ and upper limits corresponding to $t = t$ and $\mathbf{v} = \mathbf{v}$ will yield the velocity vector, $\mathbf{v}$ in terms of t (component by component time integration as in Section 11.11 B. & J.).

$$\int_{\mathbf{v}_0}^{\mathbf{v}} d\mathbf{v} = \int_0^t \left(4e^{-2t}\hat{\mathbf{i}} + 3t^2\hat{\mathbf{j}} + 6\cos(3t)\hat{\mathbf{k}}\right)dt$$

$$\mathbf{v} - \mathbf{v}_0 = \left(-2e^{-2t}\hat{\mathbf{i}} + t^3\hat{\mathbf{j}} + 2\sin(3t)\hat{\mathbf{k}}\right) - \left(-2\hat{\mathbf{i}}\right)$$

Substituting velocity vector at time $t = 0$, $\mathbf{v}_0 = 2\hat{\mathbf{j}}$ yields,

$$\mathbf{v} - 2\hat{\mathbf{j}} = \left(-2e^{-2t}\hat{\mathbf{i}} + t^3\hat{\mathbf{j}} + 2\sin(3t)\hat{\mathbf{k}}\right) - \left(-2\hat{\mathbf{i}}\right)$$

Velocity vector, $\underline{\mathbf{v} = \left(2 - 2e^{-2t}\right)\hat{\mathbf{i}} + \left(t^3 + 2\right)\hat{\mathbf{j}} + 2\sin(3t)\hat{\mathbf{k}}}$

Similarly, integrating the velocity vector, $\mathbf{v}$ with respect to t, will yield the position vector, $\mathbf{r}$ with respect to the origin, $\mathbf{r}_0$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(2 - 2e^{-2t}\right)\hat{\mathbf{i}} + \left(t^3 + 2\right)\hat{\mathbf{j}} + 2\sin(3t)\hat{\mathbf{k}}$$

$$\int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r} = \int_0^t \left[\left(2 - 2e^{-2t}\right)\hat{\mathbf{i}} + \left(t^3 + 2\right)\hat{\mathbf{j}} + 2\sin(3t)\hat{\mathbf{k}}\right]dt$$

$$\mathbf{r} - \mathbf{r}_0 = \left[\left(2t + e^{-2t}\right)\hat{\mathbf{i}} + \left(\frac{t^4}{4} + 2t\right)\hat{\mathbf{j}} - \frac{2}{3}\cos(3t)\hat{\mathbf{k}}\right] - \left(\hat{\mathbf{i}} - \frac{2}{3}\hat{\mathbf{k}}\right)$$

When t = 0, $\mathbf{r}_0 = 0$. Therefore, position vector from the origin,

$$\underline{\mathbf{r} = \left[\left(2t - 1 + e^{-2t}\right)\hat{\mathbf{i}} + \left(\frac{t^4}{4} + 2t\right)\hat{\mathbf{j}} + \frac{2}{3}\left(1 - \cos(3t)\right)\hat{\mathbf{k}}\right]}$$

3. At a general time t, a particle has position

$$\mathbf{r} = 3t\,\hat{\mathbf{i}} + 2t^3\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$$

in which **r** is in m and *t* in s. Find at *t* = 1 s, the following:

(i) velocity and acceleration vectors in Cartesian coordinates
(ii) unit tangent vector to path $\hat{\tau}$
(iii) tangential component of acceleration a_τ
(iv) normal component of acceleration a_n
(v) radius of curvature of path ρ
(vi) unit normal vector $\hat{\mathbf{n}}$

Solution:

(i) Position vector, $\mathbf{r} = 3t\,\hat{\mathbf{i}} + 2t^3\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}$

Velocity vector, $\dfrac{d\mathbf{r}}{dt} = \mathbf{v} = 3\,\hat{\mathbf{i}} + 6t^2\hat{\mathbf{j}} + 6t\,\hat{\mathbf{k}}$

Acceleration vector, $\dfrac{d\mathbf{v}}{dt} = \mathbf{a} = 12t\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}$

Therefore, at *t* = 1 s,
Position vector, $\mathbf{r} = \left(3\,\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}\right)$ m

Velocity vector, $\dfrac{d\mathbf{r}}{dt} = \mathbf{v} = \left(3\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}\right)$ m/s

Acceleration vector, $\dfrac{d\mathbf{v}}{dt} = \mathbf{a} = \left(12\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}\right)$ m/s^2

(ii) Velocity scalar, $v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{3^2 + 6^2 + 6^2} = 9\,\text{m/s}$

Unit tangent vector to path $\hat{\tau} = \dfrac{\mathbf{v}}{v} = \left(\dfrac{3}{9}\right)\hat{\mathbf{i}} + \left(\dfrac{6}{9}\right)\hat{\mathbf{j}} + \left(\dfrac{6}{9}\right)\hat{\mathbf{k}}$

$$\hat{\tau} = \frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}$$

(iii) Tangential component of acceleration $a_\tau = \mathbf{a}\cdot\hat{\tau}$

$$a_\tau = \left(12\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}\right)\cdot\left(\frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}\right)$$

$$a_\tau = \left(12\times\frac{2}{3}\right) + \left(6\times\frac{2}{3}\right)$$

$$a_\tau = 12\,\text{m/s}^2$$

(iv) Normal component of acceleration $a_n = \sqrt{|\mathbf{a}|^2 - a_t^2}$

$$a_n = \sqrt{\left(\sqrt{12^2 + 6^2}\right)^2 - 12^2}$$

$$a_n = 6\,\text{m/s}^2$$

(v) Radius of curvature of path $\rho = \dfrac{v^2}{a_n}$

$$\rho = \frac{9^2}{6}$$

$$\rho = 13.5\,\text{m}$$

(vi) Since $\mathbf{a} = a_\tau \hat{\tau} + a_n \hat{\mathbf{n}}$

Unit normal vector $\hat{\mathbf{n}} = \dfrac{\mathbf{a} - a_\tau \hat{\tau}}{a_n}$

$$\hat{\mathbf{n}} = \frac{\left(12\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}\right) - 12\left(\frac{1}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}\right)}{6}$$

$$\hat{\mathbf{n}} = -\frac{2}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} - \frac{1}{3}\hat{\mathbf{k}}$$

4. At a given instant in an airplane race, airplane A is flying horizontally in a straight line, and its speed is being increased at a rate of 6 m/s^2. Airplane B is flying at the same altitude as airplane A and, as it rounds a pylon, is following a circular path of 200-m radius. Knowing that at the given instant the speed of B is being decreased at the rate of 2 m/s^2, determine, for the positions shown, (a) the velocity of B relative to A, (b) the acceleration of B relative to A. (B. & J. 11.141)

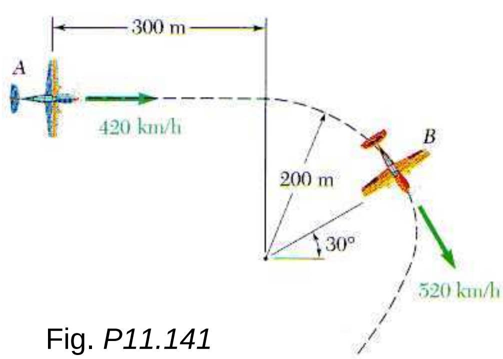


Fig. *P11.141*

Solution:

(a) Velocity of A, $\mathbf{v}_A = \left(420\,\hat{\mathbf{i}}\right)$km/h

Velocity of B, $\mathbf{v}_B = \left(520\cos 60^\circ\,\hat{\mathbf{i}} - 520\sin 60^\circ\,\hat{\mathbf{j}}\right)$km/h

Eqn. 11.33 (B. & J.)

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$
$$\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$$
$$\mathbf{v}_{B/A} = \left(520\cos 60^\circ\,\hat{\mathbf{i}} - 520\sin 60^\circ\,\hat{\mathbf{j}}\right) - 420\,\hat{\mathbf{i}}$$
$$\mathbf{v}_{B/A} = -160\,\hat{\mathbf{i}} - 450.33\,\hat{\mathbf{j}}$$
$$\left|\mathbf{v}_{B/A}\right| = \sqrt{160^2 + 450.33^2}$$
$$\left|\mathbf{v}_{B/A}\right| = 477.91 \text{ km/h}$$

and the angle between $\mathbf{v}_{B/A}$ and the *x*-axis is

$$\cos\theta = \frac{\left(v_{B/A}\right)_x}{v_{B/A}}$$
$$\cos\theta = \frac{160}{477.91}$$
$$\theta = 70.44^\circ$$

Therefore, the velocity of B relative to A, $\mathbf{v}_{B/A} = 477.91 \text{ km/h}$ 70.44°

(b) Acceleration of A, $\mathbf{a}_A = 6\,\hat{\mathbf{i}}$

Tangential acceleration component of B, $\left(\mathbf{a}_B\right)_\tau = 2\left(-\sin 30^\circ\,\hat{\mathbf{i}} + \cos 30^\circ\,\hat{\mathbf{j}}\right)$

Normal acceleration component of B, $\left(\mathbf{a}_B\right)_n = \dfrac{v_B^2}{\rho} = \dfrac{144.44^2}{200} = 104.32 \text{ m/s}^2$ 30°

$$\left(\mathbf{a}_B\right)_n = 104.32\left(-\cos 30^\circ\,\hat{\mathbf{i}} - \sin 30^\circ\,\hat{\mathbf{j}}\right)$$

Eqn. 11.34 (B. & J.)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$
$$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$$
$$\mathbf{a}_{B/A} = \left[\left(\mathbf{a}_B\right)_\tau + \left(\mathbf{a}_B\right)_n\right] - \mathbf{a}_A$$
$$\mathbf{a}_{B/A} = 2\left(-\sin 30^\circ\,\hat{\mathbf{i}} + \cos 30^\circ\,\hat{\mathbf{j}}\right) + 104.32\left(-\cos 30^\circ\,\hat{\mathbf{i}} - \sin 30^\circ\,\hat{\mathbf{j}}\right) - 6\,\hat{\mathbf{i}}$$
$$\mathbf{a}_{B/A} = \left(-97.34\,\hat{\mathbf{i}} - 50.43\,\hat{\mathbf{j}}\right) \text{ m/s}^2$$

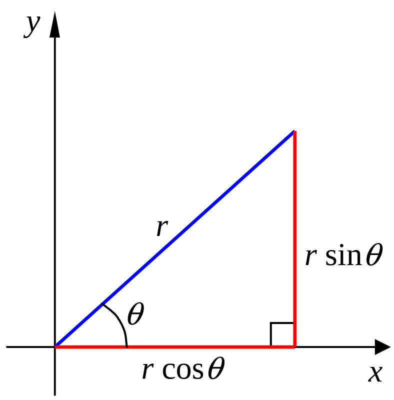
Therefore, $a_{B/A} = \sqrt{97.34^2 + 50.43^2} = 109.63 \text{ m/s}^2$

$$\tan\theta = \frac{50.43}{97.34}$$
$$\theta = 27.39^\circ$$
$$\mathbf{a}_{B/A} = 109.63 \text{ m/s}^2 \quad 27.39^\circ$$

5. The two-dimensional motion of a particle is defined by the relations $r = \dfrac{1}{\sin\theta - \cos\theta}$ and $\tan\theta = 1 + \dfrac{1}{t^2}$, where r and θ are expressed in meters and radians, respectively, and t is expressed in seconds. Determine (a) the magnitudes of the velocity and acceleration at any instant, (b) the radius of curvature of the path. What conclusion can you draw regarding the path of the particle? (B. & J. 11.166)

Solution:

The polar coordinates r and θ can be converted to the Cartesian coordinates x and y,



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\frac{y}{x} = \tan\theta$$

Therefore, the distance can be written as

$$r = \frac{1}{\sin\theta - \cos\theta}$$

Substituting $\sin\theta$ and $\cos\theta$,

$$r = \frac{1}{\dfrac{y}{r} - \dfrac{x}{r}}$$
$$= \frac{r}{y - x}$$

Thus,

$$y - x = 1$$
$$y = x + 1 \qquad (1)$$

And the angle is

$$\tan\theta = 1 + \frac{1}{t^2}$$

Substituting $\tan\theta = \dfrac{y}{x}$,

$$\frac{y}{x} = 1 + \frac{1}{t^2} \qquad (2)$$

Substituting (1) into (2), to obtain *x*-coordinate

$$\frac{x+1}{x} = 1 + \frac{1}{t^2}$$

$$x + 1 = x + \frac{x}{t^2}$$

$$\frac{x}{t^2} = 1$$

$$x = t^2$$

And the *y*-coordinate,

$$\frac{y}{x} = 1 + \frac{1}{t^2}$$

$$\frac{y}{y-1} = 1 + \frac{1}{t^2}$$

$$y = (y-1)\left(1 + \frac{1}{t^2}\right)$$

$$y = y - 1 + \frac{y}{t^2} - \frac{1}{t^2}$$

$$\frac{y}{t^2} = 1 + \frac{1}{t^2}$$

$$y = t^2 + 1$$

(a) Position vector, $\mathbf{r} = \left(t^2, t^2 + 1 \right)$

Velocity vector, $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(2t, 2t \right)$

Magnitude of velocity, $v = \sqrt{(2t)^2 + (2t)^2} = 2t\sqrt{2}$

Acceleration vector, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(2, 2 \right)$

Magnitude of acceleration, $a = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

(b) Since $y = x + 1$ is a linear equation, the radius of curvature of the path, $\rho = \infty$

The particle is moving in a straight line.